Math 6630: Numerical Solutions of Partial Differential Equations
Background and Review: PDEs

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January 11, 2023

"Prerequisites"

Several topics are background for this course:

– (Numerical) linear algebra

– Calculus

– "Basic" knowledge of ordinary/partial differential equations

– Some programming experience

We'll spend some time briefly reviewing portions of these.

Notation

We'll generally work over d-dimensional physical space, $d = 1, 2, 3$, with variables x, y, z.
We'll use t as the time variable for time-dependent problems.

Generally we'll refer to state (unknown) functions as u, e.g.,

$$u(x), \quad u(x, y), \quad u(t, x), \quad u(t, x, y, z)$$

(Partial) Derivatives are abbreviated with subscripts,

$$\frac{\partial u}{\partial t} = u_t, \qquad\qquad \frac{\partial^3 u}{\partial x^3} = u_{xxx}.$$

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Basic differential equations

Ordinary or partial differential equations (ODEs/PDEs) are mathematical laws governing an unknown u that model some phenomenon.

$$u_t = u_{xx}, \qquad \text{(Heat diffusion in 1D)}$$
$$u_{tt} = u_{xx}, \qquad \text{(Wave motion in 1D)}$$
$$u_{xx} + u_{yy} = 0, \qquad \text{(Steady-state temperature in 2D)}$$

Differential equations are incomplete knowledge of u without initial/boundary conditions.

Differential equations can have *input* parameters, e.g., a scalar coefficient or a function.

$$u_{xx} = f(x), \qquad\qquad u_t = \kappa u_{xx},$$

At a high level, one can view the task of solving a differential equation as a map from inputs (e.g., f, κ) to outputs (the solution u).

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Well-posedness

We can view the task of solving PDEs as a function from inputs to outputs:

$$\text{Inputs, e.g., } f, \kappa \xrightarrow{\text{Solution map}} \text{Solution } u, \text{ the "output"}$$

It is reasonable that we should really only try to solve a PDE if we know that the above procedure is **well-posed**.

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For example, <u>non-existence</u>:

A PDE is ill-posed if for a given input, there is no solution u. E.g.,

$$
\begin{aligned}
u_t &= u_x, & x &\in (0, 2\pi), \ t > 0 \\
u(x, 0) &= \sin x, & x &\in [0, 2\pi], \\
u(0, t) &= 0, & t &> 0 \\
u(2\pi, t) &= 0, & t &> 0.
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For example, non-uniqueness:

A PDE is ill-posed if for a given input, there are multiple solutions u. E.g.,

$$u'' = \sin x, \qquad\qquad x \in (0, 2\pi)$$
$$u'(0) = 0,$$
$$u'(2\pi) = 0.$$

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For example, <u>ill-behaved</u> properties:

A PDE is ill-posed if it depends on input parameters in "ill-behaved" ways. E.g.,

$$
\begin{aligned}
u_t &= -u_{xx}, & x &\in (0, 2\pi), \ t > 0 \\
u(x,0) &= f(x), & x &\in [0, 2\pi], \\
u(0,t) &= 0, & t &> 0 \\
u(2\pi, t) &= 0. & t &> 0
\end{aligned}
$$

Numerical methods: overall goals

"Most" PDEs *cannot* be analytically solved ☹

Our main strategy for recourse is to *approximate* the solution with a numerically computed one.

We will *always* assume that a given ODE/PDE is well-posed.
(If it's not, why bother to compute an approximate solution to a non-existent/non-unique/ill-behaved exact solution?)

For numerical methods, we typically want the following things:

– Stability: The method does not "blow up" given reasonable inputs

– Accuracy: The solution computed by the method is "close" to the exact solution.

– Efficiency: The method does not take too much computational effort to compute a solution, and/or the memory and operation complexity required to compute a solution can be estimated.

– Simple: The method can be implemented and deployed with relative ease.

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Summaries, necessarily simplistic
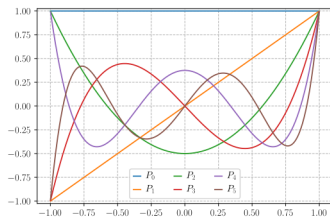
Finite difference methods:

- + Easy, simple, transparent
- − Relatively inflexible order of accuracy
- − Difficult for complex geometries
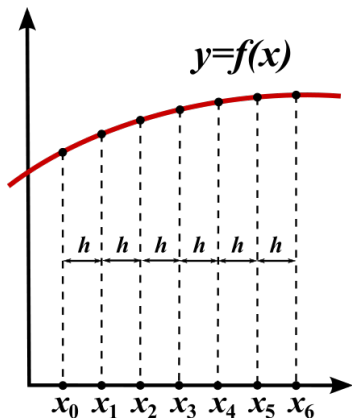
Summaries, necessarily simplistic

Fourier/spectral methods:
- + Conceptually simple
- + "Infinite order" accuracy
- − Very difficult for complex geometries
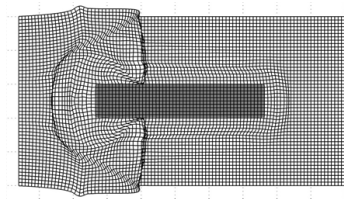- − Can suffer instability

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Summaries, necessarily simplistic

Finite volume methods:

- $+$ Solid mathematical theory
- $+$ Can model non-smooth solutions
- $-$ Low order accuracy

Summaries, necessarily simplistic

Finite element methods:

- \+ Solid mathematical theory
- \+ High-order and geometric flexibility
- − Can involve technical mathematics
- − Can be complicated to implement
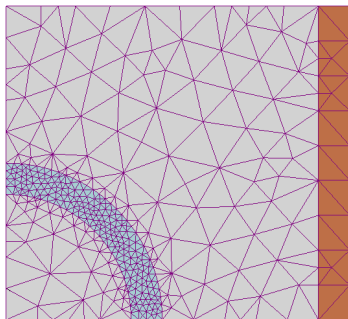
📄 LeVeque, Randall J. (2002). *Finite Volume Methods for Hyperbolic Problems*. Cambridge University Press. ISBN: 978-1-139-43418-8.

📄 — (2007). *Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-Dependent Problems*. SIAM. ISBN: 978-0-89871-783-9.