DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH
Numerical Solutions of PDE
MATH 6630 – Section 001 – Spring 2023

Project 1
Finite difference methods
Due Friday, March 3, 2021

You must complete **only one** of the three exercises below, of your choice.

Submit your project via Github in a project named `math6630-project-1` by adding me (`akilnarayan`) as a collaborator to the repository. You may set up and share the repository before the due date. I will only clone the project after the due date.

A LaTeX template (that you should use for your submission) is located at `https://github.com/akilnarayan/math6630-project-1-template`, which also contains the typesetting source for the problem statements below. You can delete the typesetting for the problems that you choose *not* to complete. You may amend the template in any reasonable way you choose (e.g., if you prefer a different style/font).

1. (Finite difference methods in 1D)
Consider the ordinary differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\kappa(x)\frac{\mathrm{d}}{\mathrm{d}x}u(x)\right) = f(x), \qquad x \in [0,1], \tag{1}$$

with homogeneous Dirichlet boundary conditions, $u(0) = u(1) = 0$, and where scalar diffusion coefficient κ is given by,

$$\kappa(x) = 2 + \sum_{\ell=1}^{5} \frac{1}{\ell+1}\sin(\ell\pi x).$$

The goal of this exercise will be to numerically compute solutions to this problem.

(a) Define the operator,

$$\widetilde{D}_0 u(x_j) = \frac{u(x_j + h/2) - u(x_j - h/2)}{h}, \qquad h = 1/(N+1), \qquad x_j := jh,$$

for a fixed number of points $N \in \mathbb{N}$. Then with u_j the numerical solution approximating $u(x_j)$ for solving the $d = 1$ version of (1), consider the scheme,

$$\widetilde{D}_0\left(\kappa(x_j)\widetilde{D}_0 u_j\right) = f(x_j), \qquad j \in [N]. \tag{2}$$

Show that, for smooth u and κ, this scheme has second-order local truncation error.

(b) Construct an exact solution via the *mathed of manufactured solutions*: posit an exact (smooth) solution $u(x)$ (that satisfies the boundary conditions!) and, compute f in (1) so that your posited solutions satifies (1).

(c) Implement the scheme above for solving (1), setting f to be the function identified in part (b), so that you know the exact solution. Show that indeed you achieve second-order convergence in h (say in the $h^{d/2}$-scaled vector ℓ^2 norm) . (To "show" this, plot on a log scale the error as a function of a discretization parameter, such as h or N, and verify that the slope of the resulting line is what is expected.)

2. (Finite difference methods in 2/3D)
Consider the following partial differential equation that generalizes (1):

$$\nabla \cdot (\kappa(\boldsymbol{x}) \nabla \boldsymbol{u}(\boldsymbol{x})) = f(\boldsymbol{x}), \qquad \boldsymbol{x} \in [0,1]^d, \tag{3}$$

again with homoegenous Dirichlet boundary conditions, $u\big|_{\partial[0,1]^d} = 0$. Set the diffusion coefficient to be,

$$\kappa(\boldsymbol{x}) = 2 + \sum_{k,\ell=1}^{3} \frac{1}{(k+1)(\ell+1)} \sin(\ell \pi x_1) \sin(k \pi x_2), \qquad \boldsymbol{x} = (x_1, x_2)^T.$$

This problem involves numerically solving the PDE above.

(a) Consider $d = 2$. To discretize the ∇ operator for $d = 2$, $\boldsymbol{x} = (x_1, x_2)^T$, use,

$$\nabla \sim \begin{pmatrix} \widetilde{D}_{0,1} \\ \widetilde{D}_{0,2} \end{pmatrix},$$

where $\widetilde{D}_{0,1}$ and $\widetilde{D}_{0,1}$ are one-dimensional versions of (2) operating in the x_1 and x_2 directions, respectively. Use the method of manufactured solutions to define an appropriate f so that you know the exact solution. Verify expected order of accuracy (say in h) as in the previous problem. What novel practical aspects arise in the two-dimensional case compared to the 1D case?

(b) Can you extend your solver to three dimensions? Do you still observe high-order convergence? Note that in either 2 or 3 dimensions, you may want to consider iterative methods for solving the linear system. (Does the matrix $\boldsymbol{A}$ in your linear system have special properties or structure?) Note also that for these problems, if $\boldsymbol{u}$ is a vector containing the degrees of freedom for the solution u, then you can evaluate $\boldsymbol{u} \mapsto \boldsymbol{A}\boldsymbol{u}$ *without* forming the full d-dimensional $\boldsymbol{A}$ matrix, and instead using only "one-dimensional" versions of $\boldsymbol{A}$.

3. (Finite difference methods for time-dependent problems)
Consider the PDE,

$$u_t + a u_x = 0, \qquad u(x,0) = \exp(\sin 2\pi x), \qquad x \in [0,1),$$

with periodic boundary conditions, where k is the timestep. In this problem, we'll use the following *Lax-Wendroff* scheme to numerically solve this PDE:

$$D^+ u_j^n = -a D_0 u_n^j + \frac{a^2 k}{2} D_+ D_- u_j^n.$$

(a) Show that this scheme has local truncation error that is order h^2 in space and k^2 in time.

(b) Compute the stability bound relating k and h via von Neumann stability analysis.

(c) Implement the Lax-Wendroff scheme (say with $a = 1$ and integrating up to time $T = 1$) and numerically verify that the scheme is second-order in space, and second-order in time.