This week: last week of new material

Next Tuesday (Apr. 27): review session.

last day of classes

Wed May 5: Final exam @ 8am.
(same format as midterms)

- Hw #9 due <u>today</u>.
- Quiz (#7?) on canvas due <u>tomorrow</u> (available now).
- Hw #10 posted, due Apr. 27 (last day of classes)
  (last one)
- Quiz (#8?) due on Apr. 27. (Available Mon. Apr. 26-27).
  (last one)

Office hours Thursday this week

- Thursday, (Apr. 22) from 2-3pm (special time)
  (on Canvas calendar)

PDEs on infinite domains

MATH 3150 Lecture ~~09~~ 10

April 13, 2021

Haberman 5th edition: Section 10.4

The Fourier transform

Given a function $f(x)$ defined on the real line, $-\infty < x < \infty$, the Fourier transform of f is defined as

$$\mathcal{F}\{f\}(\omega) = F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\mathrm{d}x, \qquad -\infty < \omega < \infty.$$

Given a function $F(\omega)$ defined on the real line, $-\infty < \omega < \infty$, the inverse Fourier transform of F is defined as

$$\mathcal{F}^{-1}\{F\}(x) = f(x) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\mathrm{d}\omega, \qquad -\infty < x < \infty.$$

if f is smooth, then $\mathcal{F}^{-1}\{\mathcal{F}\{f\}\} = f$.

We will now use the Fourier transform to solve PDEs on infinite domains.

The heat equation

Using the Fourier transform, compute the solution to the PDE,

$$u_t = k\, u_{xx}, \qquad t > 0, \quad -\infty < x < \infty$$
$$u(x,0) = f(x). \qquad k > 0 \text{ given}$$

$$u_t = k u_{xx} \xrightarrow[\text{in } x\text{-space}]{\mathcal{F}} \mathcal{F}\left\{\frac{\partial}{\partial t} u\right\} = k\, \mathcal{F}\{u_{xx}\}$$

$\Big\downarrow$ $\frac{\partial}{\partial t}$ is "independent" of $\mathcal{F}$ in x

$$\frac{\partial}{\partial t} \mathcal{F}\{u\} = k\, \mathcal{F}\{u_{xx}\}$$

if $U(\omega,t) = \mathcal{F}\{u(x,t)\}$, then $\mathcal{F}\{u_{xx}\} = (-i\omega)^2\, U(\omega,t)$

$$\frac{\partial}{\partial t} U(\omega,t) = (-i\omega)^2 K\, U(\omega,t)$$

$$U_t = -k\omega^2 U$$

$$u(x,0)=f(x) \xrightarrow{\mathcal{F}} U(\omega,0)=F(\omega)$$

Summary: $\left.\begin{array}{l} \frac{\partial}{\partial t} U(\omega,t) = -k\omega^2 U \\ U(\omega,0)=F(\omega) \end{array}\right\}$ ordinary diff. eqn. (initial value problem)

treat ω as an independent parameter.

$U_t = -k\omega^2 U$ (compare to $y' = ay$)

$U(\omega,t) = C \exp(-k\omega^2 t)$

(from ODE's: separable eqn, constant coeff homogeneous eqn, exact equation)

<u>Note</u>: C is a constant with respect to t.
<u>But</u> it can depend on ω. $C = C(\omega)$

General sol'n: $U(\omega,t) = C(\omega)\exp(-k\omega^2 t)$

$t=0$ $U(\omega,0)=F(\omega)$

$U(\omega,0) = C(\omega) \implies C(\omega) = F(\omega)$

PDE solution (in freq. space): $U(\omega,t) = F(\omega)\exp(-k\omega^2 t)$

in physical space?

Define $G(\omega,t) = \exp(-kt\omega^2) \Longrightarrow g(x,t) = \sqrt{\frac{\pi}{kt}} \cdot \exp\left(\frac{-x^2}{4kt}\right)$

formula sheet
$a = kt$

$U(\omega,t) = F(\omega)\, G(\omega,t)$

$\Longrightarrow u(x,t) = (f * g)(x,t)$

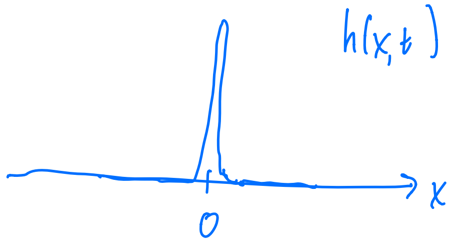
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\, g(x-s,t)\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\sqrt{\frac{\pi}{kt}}\, \exp\left(\frac{-(x-s)^2}{4kt}\right) ds$$

$$u(x,t) = \int_{-\infty}^{\infty} f(s)\sqrt{\frac{1}{4\pi kt}}\, \exp\left(-\frac{(x-s)^2}{4kt}\right) ds$$

Define $h(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{x^2}{4kt}\right)$

$0 < t$ small



t large

$h(x,t)$ is a Gaussian, whose width increases in time.

The heat kernel, I

The function,

$$h(x,t) = \frac{1}{\sqrt{4\pi k t}} \exp\left(-\frac{x^2}{4kt}\right)$$

is called the *heat kernel*.

The heat kernel, I

The function,

$$h(x,t) = \frac{1}{\sqrt{4\pi k t}} \exp\left(-\frac{x^2}{4kt}\right)$$

is called the *heat kernel*.

From the previous example, the solution to the heat equation is simply written:

$$u(x,t) = \big(f(x) * h(x,t)\big) \div 2\pi,$$

where the convolution is taken over the x variable.

The heat kernel, II

Note that the heat kernel is actually a particular solution to the heat equation.

Example

Show that the solution $u(x,t)$ to $u_t = ku_{xx}$ with initial data $u(x,0) = \delta(x)$ is the heat kernel $u(x,t) = h(x,t)$.

Dirac delta fcn.

$$u_t = ku_{xx} \xrightarrow{\mathcal{F}} U_t = k(-i\omega)^2 U$$

$$= -k\omega^2 U$$

$$U(\omega,t) = C(\omega)\exp(-k\omega^2 t)$$

$$U(\omega,0) = ?$$

$$U(w,0) = \mathcal{F}\{\delta(x)\} = \frac{1}{2\pi}$$

$$\Longrightarrow U(w,0) = C(w) = \frac{1}{2\pi}$$

$$U(w,t) = \frac{1}{2\pi} \exp(-kw^2 t)$$

$$u(x,t) = \frac{1}{2\pi}\sqrt{\frac{\pi}{kt}} \exp\left(\frac{-x^2}{4kt}\right)$$

$$= \frac{1}{\sqrt{4\pi kt}} \exp\left(-x^2/4kt\right) = h(x,t) \checkmark$$

The heat kernel, II

Note that the heat kernel is actually a particular solution to the heat equation.

Example

Show that the solution $u(x,t)$ to $u_t = ku_{xx}$ with initial data $u(x,0) = \delta(x)$ is the heat kernel $u(x,t) = h(x,t)$.

The heat kernel is an example of a broader class of solutions.

The heat kernel, III

Suppose L is a linear differential operator (in both x and t), and that L is first-order in t.
Let $q(x,t)$ be the solution to the PDE with Dirac mass initial data,

$$\begin{aligned} Lq &= 0, && t > 0, \ \ -\infty < x < \infty \\ q(x,0) &= \delta(x). \end{aligned}$$

Such solutions q are also sometimes called *fundamental solutions* or *impulse responses*.

If $L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$, then $q(x,t)$ is the heat kernel $h(x,t)$.

The heat kernel, III

Suppose L is a linear differential operator (in both x and t), and that L is first-order in t.
Let $q(x,t)$ be the solution to the PDE with Dirac mass initial data,

$$\begin{aligned} Lq &= 0, & t > 0,\ \ -\infty < x < \infty \\ q(x,0) &= \delta(x). \end{aligned}$$

Such solutions q are also sometimes called *fundamental solutions* or *impulse responses*.

If $L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$, then $q(x,t)$ is the heat kernel $h(x,t)$.

Example

With the notation above, show that the solution u to the PDE

$$\begin{aligned} Lu &= 0, & t > 0,\ \ -\infty < x < \infty \\ u(x,0) &= f(x) \end{aligned}$$

is given by $u = f * q$, where the convolution is taken over the x variable.

- HW #10 due Tuesday
- Quiz due on Tuesday (Canvas)
- Tuesday is a review session (office hours)
- Office hours today are a special time: 2-3pm
- Final exam: Wed May 5 @ 8am (till 10am)
  - formula sheet on web/Canvas (3 pages)
  - is cumulative (covers all homework material)

The wave equation

Using the Fourier transform, compute the solution to the PDE,

$$u_{tt} = c^2 u_{xx}, \qquad t > 0, \quad -\infty < x < \infty$$

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x).$$

The wave equation

Using the Fourier transform, compute the solution to the PDE,

$$u_{tt} = c^2 u_{xx}, \qquad t > 0, \quad -\infty < x < \infty \qquad c \text{ is given}$$

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x).$$

Specialize the solution above to the case $g = 0$.

Assume $g = 0$ to start.

$$u_{tt} = c^2 u_{xx} \xrightarrow{\mathcal{F} \text{ (in } x)} U_{tt} = c^2(-i\omega)^2 U, \quad U = U(\omega, t)$$

$$u(x,0) \xrightarrow{\mathcal{F}} U(\omega, 0) = F(\omega)$$

$$u_t(x,0) \xrightarrow{\mathcal{F}} U_t(\omega, 0) = G(\omega) = 0$$

$$\left.\begin{aligned} &U_{tt} + c^2 w^2 U = 0 \\ &U(w,0) = F(w) \\ &U_t(w,0) = 0 \end{aligned}\right\}$$ ODE
linear, homogeneous, constant-coeff.

characteristic eqn roots: $r = \pm icw$ $\quad (r^2 + c^2w^2 = 0)$

$$U(w,t) = A(w)\cos(wct) + B(w)\sin(wct)$$

initial data: $U(w,0) = A(w) = F(w) \implies A = F$

$$U_t(w,0) = wc\,B(w) = 0 \implies B = 0.$$

$$U(w,t) = F(w)\cos(wct)$$

$$\mathcal{F}^{-1}\{F(w)\cos(wct)\} = ?$$

recall: $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

(from $e^{i\theta} = \cos\theta + i\sin\theta$)

$$F(w)\cos(wct) = \tfrac{1}{2}F(w)e^{iwct} + \tfrac{1}{2}F(w)e^{-iwct}$$

$$\Longrightarrow \quad u(x,t) = \mathcal{F}^{-1}\{U(\omega,t)\}$$

$$= \tfrac{1}{2}\mathcal{F}^{-1}\{F(\omega)e^{i\omega ct}\} + \tfrac{1}{2}\mathcal{F}^{-1}\{F(\omega)e^{-i\omega ct}\}$$

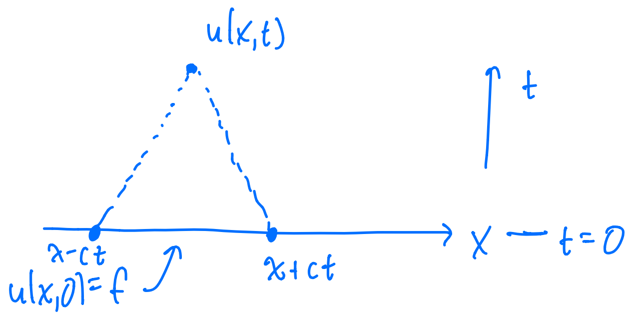
$$u(x,t) = \tfrac{1}{2} f(x-ct) + \tfrac{1}{2} f(x+ct)$$

recall: $u(x,0) = f(x)$

$f(x-ct)$: $f(x)$ shifted ct units to the right

$f(x-ct)$: wave moving to the right with speed c.

$f(x+ct)$: " " " " left with speed c.



Solutions depends on initial data at $x \pm ct$.

"Information" travels at a finite speed c.

what if $g(x) \neq 0$? $\quad u_t(x,0) = g(x)$

Most things are unchanged : $\quad U_{tt} + c^2\omega^2 U = 0$

$$U(\omega, \cancel{t}\,0) = F(\omega)$$

$$U_t(\omega, \cancel{t}\,0) = G(\omega)$$

$$U(w,t) = A(w)\cos(wct) + B(w)\sin(wct)$$

$$U_t(w,t) = -wc\,A(w)\sin(wct) + wc\,B(w)\cos(wct)$$

$$\Longrightarrow A(w) = F(w), \quad B(w) = \frac{G(w)}{wc}$$

$$U(w,t) = \underbrace{F(w)\cos(wct)}_{\text{we already know how to inverse transform this}} + \underbrace{\frac{G(w)}{wc}\sin(wct)}_{\text{focus on this.}}$$

$$\mathcal{F}^{-1}\left\{\frac{G(w)}{wc}\sin(wct)\right\} = ?$$

since $\sin(wct) = \frac{1}{2i}\left[e^{iwct} - e^{-iwct}\right]$

$$\frac{G(w)}{wc}\sin(wct) = \frac{1}{2c}\frac{G(w)}{iw}e^{iwct} - \frac{1}{2c}\frac{G(w)}{iw}e^{-iwct}$$

Consider $\mathcal{F}^{-1}\left\{\frac{G(w)}{iw}e^{iwct}\right\}$

From the shift property: this equals the inverse transform of $\frac{G(w)}{iw}$ evaluated at $x-ct$

What is $\mathcal{F}^{-1}\left\{\frac{G(w)}{iw}\right\}$? formula sheet

Define $H(w) = \frac{G(w)}{iw}$, $h(x) = \mathcal{F}^{-1}\{H\}$

Note: $\mathcal{F}^{-1}\{iw \cdot H(w)\} = -\frac{d}{dx} h(x)$

$$\begin{gathered} \mathcal{F}^{-1}\{iw \cdot H(w)\} \\ \| \\ \mathcal{F}^{-1}\{G\} = g(x) \end{gathered}$$

$$\Longrightarrow h(x) = -\int^{x} g(s)\,ds + K_1, \quad K_1 \text{: unknown}$$

$$\Longrightarrow \mathcal{F}^{-1}\left\{\frac{G(w)}{iw} e^{iwct}\right\} = h(x-ct)$$

$$= -\int^{x-ct} g(s)\,ds + K_1$$

and $\mathcal{F}^{-1}\left\{\frac{G(w)}{iw} e^{-iwct}\right\} = -\int^{x+ct} g(s)\,ds + K_2$

Putting it all together:

$$\underbrace{G(w)}_{wc} \sin(wct) = \frac{1}{2c}\frac{G(w)}{iw} e^{iwct} - \frac{1}{2c}\frac{G(w)}{iw} e^{-iwct}$$

$$\Rightarrow \mathcal{F}^{-1}\left\{ \frac{G(\omega)}{\omega c} \sin(\omega c t) \right\}$$

$$= \frac{-1}{2c} \int^{x-ct} g(s)\, ds + \frac{K_1}{2c}$$

$$+ \frac{1}{2c} \int^{x+ct} g(s)\, ds - \frac{K_2}{2c}$$

$$= K + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds$$

unknown constant

$$u(x,t) = \mathcal{F}^{-1}\{ U(\omega, t) \}$$

$$= \mathcal{F}^{-1}\{ F(\omega) \cos(\omega c t) \} + \mathcal{F}^{-1}\left\{ \frac{G(\omega)}{\omega c} \sin(\omega c t) \right\}$$

$$= \frac{1}{2} f(x-ct) + \frac{1}{2} f(x+ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds + K$$

$K = ?$

$u(x,0) = f(x)$

$u(x,0) = \frac{1}{2} f(x) + \frac{1}{2} f(x) + \cancelto{0}{\frac{1}{2c} \int_x^x g(s)\,ds} + K$

$= f(x) + K$

$\Rightarrow K = 0$

$$\boxed{u(x,t) = \frac{1}{2} f(x-ct) + \frac{1}{2} f(x+ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\,ds}$$

"D'Alembert's Formula"

Other PDEs

Using the Fourier transform, compute the solution to the PDE,

$$u_t = c\,u_x, \qquad t > 0, \;\; -\infty < x < \infty$$
$$u(x,0) = f(x),$$

c given

Fourier transform: $U_t = -i\omega c\, U$

$U(\omega,0) = F(\omega)$

$U(\omega,t) = A(\omega)\, e^{-i\omega c t}$ (compare $y' = by$)

$U(\omega,0) = F(\omega) \Longrightarrow A(\omega) = F(\omega)$

$U(\omega,t) = F(\omega)\, e^{-i\omega c t}$

$$u(x,t) = \mathcal{F}^{-1}\{F(w)\, e^{-iwct}\}$$

$$= f(x - -ct) = \boxed{f(x+ct) = u(x,t)}$$

This is a wave moving to the left with speed c.

$u_t = c u_x$: "one-sided" wave equation.