Fourier transform properties

MATH 3150 Lecture 09

April 13, 2021

Haberman 5th edition: Sections 10.3, 10.4

The Fourier transform

Given a function $f(x)$ defined on the real line, $-\infty < x < \infty$, the Fourier transform of f is defined as

$$\mathcal{F}\{f\}(\omega) = F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\mathrm{d}x, \qquad -\infty < \omega < \infty.$$

Given a function $F(\omega)$ defined on the real line, $-\infty < \omega < \infty$, the inverse Fourier transform of F is defined as

$$\mathcal{F}^{-1}\{F\}(x) = f(x) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\mathrm{d}\omega, \qquad -\infty < x < \infty.$$

We will spend some time learning about properties of this transform.

Gaussian invariance

A function of the form $f(x) = \exp(-x^2)$ is called a *Gaussian*.

We've seen that

$$\mathcal{F}\left\{\exp\left(-\frac{x^2}{4\beta}\right)\right\} = \sqrt{\frac{\beta}{\pi}}\exp(-\beta\omega^2).$$

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Thus, the Fourier transform of a Gaussian is another Gaussian.

Moreover,

- functions that are very concentated in physical space are spread out in frequency space
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Shift theorems

From homework #8: Given f with $\mathcal{F}\{f\} = F$,

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Example

Given f, compute $\mathcal{F}\{f(x)e^{i\beta x}\}$ in terms of F, the Fourier transform of f.

I.e.:

- shifts in frequency space correspond to multiplication by complex exponential in physical space
- shifts in physical space correspond to multiplication by complex exponential in frequency space

Differentiation

What does differentiation look like in frequency space?

First easy case: if $f = f(x,t)$ has a Fourier transform with respect to the x variable,

$$\mathcal{F}\{f\} = F(\omega, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i\omega x} \mathrm{d}x,$$

then

$$\mathcal{F}\left\{\frac{\partial f}{\partial t}\right\} = \frac{\partial F}{\partial t}.$$

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Example

If $f = f(x)$, compute $\mathcal{F}\{f'\}$ in terms of F.

Thus,

- differentiation in physical space corresponds to multiplication by ω in frequency space
- differentiation in frequency space corresponds to multiplication by x in physical space

What happens for higher-order derivatives?

The Dirac delta function

We introduce the Dirac delta function or Dirac mass, $\delta(x)$.

Informally, this is often introduced as

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Example

Compute the Fourier transform of $\delta(x - x_0)$, where x_0 is a real number.

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Thus,

- Dirac masses in physical space correspond to complex exponentials in frequency space
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Convolution, I

The final and perhaps most technical property of Fourier transforms answers the following question:

If (f, F) and (g, G) are Fourier transform pairs, then what is the inverse Fourier transform of $F(\omega)G(\omega)$?

I.e., what does multiplication in frequency space correspond to in physical space?

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I.e., what does multiplication in frequency space correspond to in physical space?

Example

Compute $\mathcal{F}^{-1}(FG)$ in terms of f, g.

Convolution, II

This motivates the following definition:

Definition

Let functions f and g be given. The *convolution* of f and g is the function h defined as

$$h(x) = (f * g)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(s) f(x - s) \mathrm{d}s$$

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Example

Compute the inverse Fourier transform of $F(\omega)e^{i\beta\omega}$ in terms of f using convolutions.