# The Fourier transform and its properties

MATH 3150 Lecture 08

April 6, 2021

Haberman 5th edition: Sections 10.1 - 10.3

# PDE's on infinite domains

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Goal for the rest of the semester: solve PDEs on *unbounded* domains, e.g.,

$$u_t = u_{xx}, \qquad\qquad -\infty < x < \infty.$$

The ideas for bounded domains will extend almost directly to unbounded domains, but the language will look rather different.

## The essential change

The main difference on unbounded domains is: we will exchange a Fourier *Series* for a Fourier *Transform*.

In practice, this replaces summations by integration. Given a funciton $f(x)$,

$$\text{Fourier Series} = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right),$$

$$\text{Fourier Transform} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{i\omega x} dx.$$

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- The series is determined by the frequency coefficients $a_n$, $b_n$. The transform is determined by the frequency function $F(\omega)$.
- Series: the parameter $n$ is frequency (is discrete). Transform: the parameter $\omega$ is frequency (is continuous).
- Series: summation. Transform: integration.

## The next few weeks

Rough outline of next few weeks:

- (1.5 classes) derive relationship between Fourier series and Fourier transform
- (2.5 classes) explore Fourier transform properties
- (2 classes) use Fourier transforms to solve PDEs.

# Fourier Series $\longrightarrow$ Fourier transform

There are two ways we'll consider to make the connection between a series and a transform.

First method: via a PDE.

$$\begin{aligned} u_t &= u_{xx}, & -\infty < x < \infty \\ \lim_{x\to\infty} |u(x,t)| &= 0, & t \geqslant 0 \end{aligned}$$

What are the eigenvalues for this problem?

# Fourier Series $\longrightarrow$ Fourier transform

Second method: directly from Fourier series on $[-L, L]$

$$FS(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

What happens as $L \uparrow \infty$?

# The Fourier transform, I

Either method we have discussed results in the following definition:

### Definition

Given a function $f(x)$, the *Fourier transform* of $f$ is $F(\omega)$, defined as

$$F(\omega) = \mathcal{F}\{f\}(\omega) = \frac{1}{2\pi} f(x) e^{i\omega x} \mathrm{d}x, \qquad -\infty < \omega < \infty$$

Given a function $F(\omega)$, the *inverse Fourier transform* of $F$ is $f(x)$, defined as

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Like in the Fourier series case, it need not be the case that $\mathcal{F}^{-1}\{\mathcal{F}\{f\}\} = f$. In general:

$$\frac{1}{2}\left[f(x^+) + f(x^-)\right] = \mathcal{F}^{-1}(\mathcal{F}(f)).$$

# The Fourier transform, II

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2. Fourier Series: The series is formed by summing components over all frequencies. Fourier Transform: The inverse transform is formed by integrating components over all frequences.
3. Fourier series: applies over a bounded domain. Fourier Transform: applies over an infinite domain.

# Fourier transform examples

## Example

Compute the Fourier transform of $f(x) = \exp(-|x|)$.

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Let $\beta > 0$ be given. Show that the Fourier transform of $f(x) = \exp(-x^2/(4\beta))$ is $F(\omega) = \sqrt{\frac{\beta}{\pi}} \exp(-\beta\omega^2)$.