Laplace's equation

MATH 3150 Lecture 05

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Haberman 5th edition: Section 2.5

The heat equation

The heat equation for $u(x,t)$ is

$$u_t = k\,u_{xx}, \qquad u(x,0) = f(x),$$
$$u(0,t) = 0, \qquad u(L,t) = 0,$$

for $0 \leqslant x \leqslant L$ and $t \geqslant 0$.

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for $0 \leqslant x \leqslant L$ and $t \geqslant 0$.

What about for domains that are not (essentially) one-dimensional?

The two-dimensional heat equation

For a two-dimensional rectangle, $0 \leqslant x \leqslant L$ and $0 \leqslant y \leqslant H$, the heat equation has a similar form,

$$u_t = k\,u_{xx} + k\,u_{yy}, \qquad\qquad u(x,y,0) = f(x,y),$$

where $u(x,y,t)$ represents the temperature at location (x,y) at time t.

We also need boundary conditions on the four sides of the rectangle, e.g.,:

$$\begin{aligned} u(x,0) &= f_1(x), & u(x,H) &= f_2(x) \\ u(0,y) &= g_1(y), & u(L,y) &= g_2(x) \end{aligned}$$

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Recall that, in two dimensions, we write $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

The operator Δ is the *Laplacian* operator.

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This is *Laplace's equation*, dictating the steady-state temperature that evolves according to the heat equation.

Unlike the 1D case, solving for equilibrium solutions is much harder in 2D.

However, it can be tackled using separation of variables, with some minor modifications.

Examples, I

Example

Compute the solution $u(x, y)$ to the following PDE:

$$
\begin{aligned}
u_{xx} + u_{yy} &= 0, & 0 < x < L,\ 0 < y < H, \\
u(x, 0) &= 0, & u(x, H) = 0 \\
u(0, y) &= 0, & u(L, y) = g(y).
\end{aligned}
$$

Examples, II

Example

Compute the solution $u(x,y)$ to the following PDE:

$$
\begin{aligned}
u_{xx} + u_{yy} &= 0, & 0 < x < L,\ 0 < y < H, \\
\frac{\partial u}{\partial y}(x,0) &= 0, & \frac{\partial u}{\partial y}(x,H) = 0 \\
u(0,y) &= f(x), & \frac{\partial u}{\partial y}(L,y) = g(y).
\end{aligned}
$$