Midterm 1 **practice** **Name:** ______________________________
MATH3150, Section 02 February 5, 2021

This test is:

- closed-book
- closed-notes
- no-calculator
- 80 minutes

Indicate your answers clearly, and show your work. Partial credit will be awarded based on work shown. Full credit will not be awarded without some work shown.

Fun fact of life: if your work is not legible, I will not be able to read it. The ramifications of this outcome should be clear.

There are 3 questions with multiple parts; each question is worth a total of 20 points.

All pages are one-sided. If on any problem you require more space, use the back of the page.

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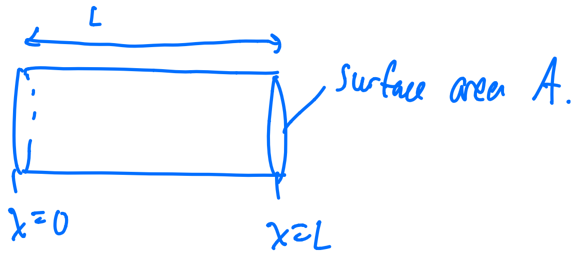
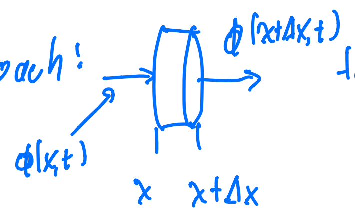
1. (20 pts) Derive the heat equation for a one-dimensional rod of length L assuming constant thermal properties and no sources. (You may ignore any initial and boundary conditions.)



$e(x,t)$: energy density.

"differential" approach:



total energy in slice:

$$\int_{slice} e(x,t)\,dx\,dy\,dz$$

$$= A\int_{x}^{\Delta x} e(x,t)\,dx$$

$$\simeq A\Delta x\, e(x,t)$$ ($e(x,t)$ is approx. constant between x and $x+\Delta x$).

time rate of change of energy in slice:

$$\frac{\partial}{\partial t}\left(A\Delta x\, e(x,t)\right)$$

$$= A\Delta x\,\frac{\partial e}{\partial t}.$$

this should equal the rate of energy change due to heat flux in/out of slice.

$\phi(x,t)$: heat flux

energy change due to heat flow/flux: $+A\,\phi(x,t) - A\phi(x+\Delta x,t)$

equating these: $A\Delta x\,\frac{\partial e}{\partial t} = A\,\phi(x,t) - A\,\phi(x+\Delta x,t)$

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$$\frac{\partial e}{\partial t} = -\left(\frac{\phi(x+\Delta x,t) - \phi(x,t)}{\Delta x} \right)$$

$$\xrightarrow{\Delta x \downarrow 0} \; = \; -\frac{\partial \phi}{\partial x}$$

1.) Relate e to temperature u:

$$\underbrace{e(x,t)}_{\text{energy density}} = \underbrace{c}_{\text{specific heat}} \cdot \underbrace{\rho}_{\text{mass density}} \cdot \underbrace{u(x,t)}_{\text{temp.}}$$

units: $\frac{\text{energy}}{\text{vol.}} = \frac{\text{energy}}{\text{mass} \cdot \text{temp}} \times \frac{\text{mass}}{\text{vol}} \times \text{temp.}$

Constant properties $\Rightarrow c, \rho$ are constant.

2.) Relate ϕ to u

Fourier's Law: "heat flows in the opposite direction of temp. gradient."

$$\phi(x,t) = -\underbrace{K}_{\text{thermal conductivity (constant)}} \frac{\partial u}{\partial x}$$

Put it all together:

$$\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x}$$

$$\frac{\partial}{\partial t}\left(c \cdot \rho \cdot u(x,t)\right) = -\frac{\partial}{\partial x}\left(-K \frac{\partial u}{\partial x}\right)$$

$$\frac{\partial u}{\partial t} = \underbrace{\frac{K}{c\rho}}_{k := K/c\rho > 0,} \frac{\partial^2}{\partial x^2} u$$

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

2. (20 pts) Solve the following eigenvalue problem: find all eigenvalues λ and eigenfunctions $\phi(x)$. You must show all work, including exhausting all possible values of λ.

$$\begin{aligned} \phi''(x) + \lambda\phi(x) = 0, &\qquad 0 < x < L \\ \phi(0) = 0, &\qquad \phi'(L) = 0 \end{aligned}$$

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3. (20 pts) Compute the solution $u(x,t)$ to the following one-dimensional heat equation:

$$u_t = u_{xx}, \qquad 0 < x < 1, \quad t > 0$$

subject to the initial and boundary conditions

$$\begin{aligned} u(x,0) &= 1 - (1-x)^2, \\ u(0,t) &= 0, \\ \frac{\partial u}{\partial x}(1,t) &= 0 \end{aligned}$$

Show all work. Your solution must be written down in terms of explicit, computable expressions or integrals, but you need not compute the values of these integrals. You may use any results derived from previous problem(s).

Separation of variables.

ansatz: $u(x,t) = \phi(x)\,T(t)$

$\Rightarrow u_t = T'(t)\,\phi(x)$

$u_{xx} = T(t)\,\phi''(x)$

$u_t = u_{xx} \longrightarrow T'(t)\,\phi(x) = T(t)\,\phi''(x)$

$\Downarrow$

$\dfrac{T'(t)}{T(t)} = \dfrac{\phi''(x)}{\phi(x)} = -\lambda$ (unknown)

$\Rightarrow T'(t) + \lambda T(t) = 0$

$\phi''(x) + \lambda\,\phi(x) = 0$

<u>BC's</u>: $u(0,t) = 0 \Rightarrow \phi(0)\,T(t) = 0 \rightarrow \phi(0) = 0$

$(T(t) = 0 \rightarrow \text{trivial})$

$$\frac{\partial u}{\partial x}(1,t)=0 \Rightarrow \phi'(1)\,T(t)=0 \rightarrow \phi'(1)=0$$

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$$T'(t)+\lambda T(t)=0$$

$$\phi''(x)+\lambda\phi(x)=0$$

$$\phi(0)=0$$

$$\phi'(1)=0$$

Solve eigenvalue problem:

$$\phi''(x)+\lambda\phi(x)=0$$

$$\phi(0)=0, \quad \phi'(1)=0$$

$\underline{\lambda<0}$: char. eqn: $r^2+\lambda=0$

$$r=\pm\sqrt{-\lambda}=\pm\sqrt{|\lambda|}, \text{ distinct}$$

$$\phi(x)=c_1\exp\left(x\sqrt{|\lambda|}\right)+c_2\exp\left(-x\sqrt{|\lambda|}\right)$$

$$\phi(0)=0 \Rightarrow c_1+c_2=0 \longrightarrow c_1=-c_2$$

$$\phi'(1)=0 \Rightarrow c_1\sqrt{|\lambda|}\exp\left(\sqrt{|\lambda|}\right)-c_2\sqrt{|\lambda|}\exp\left(-\sqrt{|\lambda|}\right)=0$$

$$c_1\sqrt{|\lambda|}\exp(\sqrt{|\lambda|})+c_1\sqrt{|\lambda|}\exp(-\sqrt{|\lambda|})=0$$

Let $z = \exp(\sqrt{|\lambda|})$

$$c_1\sqrt{|\lambda|}\left(z-\frac{1}{z}\right)=0$$

$z-\frac{1}{z}\overset{?}{=}0 \to$ not possible

$\Rightarrow z=1 \Rightarrow \sqrt{|\lambda|}=0$

$\Rightarrow c_1=0$

$\Rightarrow c_2=0$

$\Rightarrow \phi(x)=0$ is only solution.

$\lambda<0$: no eigenvalues.

$\underline{\lambda=0}$: $r^2+\lambda=0$

$r=0$ (repeated)

$$\phi(x)=c_1\exp(0x)+c_2x\cdot\exp(0\cdot x)$$

$$=c_1+c_2X$$

$\phi(0)=0 \to c_1=0$

$\phi'(1)=0 \to c_2=0$

$\to \phi(x)=0$ is only solution

$\lambda=0$ not an eigenvalue.

$\underline{\lambda > 0}$: $\quad r^2 + \lambda = 0$

$$r = \pm\sqrt{-\lambda} = \pm i\sqrt{\lambda}$$

$$\phi(x) = c_1 \cos(x\sqrt{\lambda}) + c_2 \sin(x\sqrt{\lambda})$$

$$\phi'(x) = -c_1\sqrt{\lambda}\sin(x\sqrt{\lambda}) + c_2\sqrt{\lambda}\cos(x\sqrt{\lambda})$$

$$\phi(0) = 0 \longrightarrow c_1 = 0$$

$$\phi'(1) = 0 \longrightarrow c_2\sqrt{\lambda}\cos\sqrt{\lambda} = 0$$

cosine vanishes at $\sqrt{\lambda} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots$

$$= \frac{(2n-1)\pi}{2}, \quad n = 1, 2, 3 \ldots$$

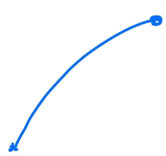
$$\lambda = \lambda_n = \left(\frac{(2n-1)\pi}{2}\right)^2$$

$\Rightarrow$ choose $c_2 = 1$, $c_1 = 0$

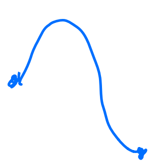
$$\phi(x) = \phi_n(x) = \sin(x\sqrt{\lambda_n})$$

$$= \sin\left(\frac{(2n-1)\pi x}{2}\right)$$

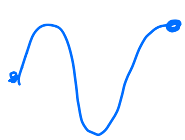
n=1



n=2



n=3



sol'n to eigenvalue problem:

$$\lambda_n = \left(\frac{(2n-1)}{2}\pi\right)^2, \quad \phi_n(x) = \sin(x\sqrt{\lambda_n})$$

Superposition $\Rightarrow$ general solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$$

$$= \sum_{n=1}^{\infty} c_n \varphi_n(x) T_n(t),$$

where $T_n(t)$ solves $T_n' + \lambda_n T_n = 0$

$\rightarrow T_n(t) = \exp(-\lambda_n t)$

char. eqn: $r + \lambda_n = 0 \rightarrow r = -\lambda_n$

$(T_n(t) = \exp(rt))$

$T_n(t) = b_n \exp(-\lambda_n t)$

any const., could depend on n.

$$u_n(x,t) = \phi_n(x)\,T_n(t)$$
$$= b_n \exp(-\lambda_n t)\sin(x\sqrt{\lambda_n})$$

Superposition:

$$u(x,t) = \sum_{n=1}^{\infty} a_n u_n(x,t) \quad (a_n \text{ arbitrary})$$
$$= \sum_{n=1}^{\infty} (a_n b_n)\exp(-\lambda_n t)\sin(x\sqrt{\lambda_n})$$

both a_n, b_n are arbitrary.

call $a_n b_n = c_n$

$$= \sum_{n=1}^{\infty} c_n \exp(-\lambda_n t)\sin(x\sqrt{\lambda_n})$$

general solution

To compute c_n: IC's + orthogonality

$$\phi_n(x) = \sin(x\sqrt{\lambda_n}) = \sin\left(\frac{(2n-1)\pi}{2} x\right)$$

$\longrightarrow$ formula sheet $\xrightarrow{L=1}$ $\displaystyle\int_0^L \phi_n(x)\phi_m(x)\,dx$

$$= \begin{cases} L/2, & n = m \geq 1 \\ 0, & n \neq m. \end{cases}$$

enforce $u(x,0) = 1-(1-x^2)^2$

$$\| $$

$$\sum_{n=1}^{\infty} c_n \phi_n(x)$$

multiply by $\phi_m(x)$
integrate over $[0,1]$

$$\int_0^1 \phi_m(x)\left[1-(1-x^2)^2\right] dx = \sum_{n=1}^{\infty} c_n \underline{\int_0^1 \phi_n(x)\phi_m(x)\,dx}$$

$L=1$ $\quad \begin{cases} \cancel{L}/2 \;1 & \text{iff } n=m \\ 0, & \text{else} \end{cases}$

$$\int_0^1 \phi_m(x)\left[1-(1-x^2)^2\right]dx = \cancel{L}/2 \;1 \; c_m$$

Sol'n: $\boxed{\begin{aligned} &u(x,t)=\sum_{n=1}^{\infty} c_n \exp(-\lambda_n t)\,\phi_n(x), \\ &\text{where } \lambda_n = \left(\frac{(2n-1)\pi}{2}\right)^2, \quad n=1,2,\ldots \\ &\phi_n(x)=\sin(x\sqrt{\lambda_n}) \\ &c_n = 2\int_0^1 \phi_n(x)\left[1-(1-x^2)^2\right]dx \end{aligned}}$

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3 'things' for exam

- derive heat eqn.
- find/compute equilibrium sol'ns
- solve heat equation