Separation of Variables

MATH 3150 Lecture 04

February 4, 2021

Haberman 5th edition: Sections 2.3-2.4

PDEs and the heat equation

Consider the following PDE problem for $u(x,t)$:

$$\begin{aligned} u_t &= k\, u_{xx}, & u(x,0) &= f(x), \\ u(0,t) &= 0, & u(L,t) &= 0, \end{aligned}$$

for $0 \leqslant x \leqslant L$ and $t \geqslant 0$.

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Our goal for the next 2 weeks is to show how to solve equations like the above one.

The technique we will use for this is *Separation of Variables*.

Separation of variables

Separation of variables has three (major) steps:

1. "Separate variables"
   - Use an educated guess to turn PDEs into ODEs
   - Rewrite PDE boundary conditions as ODE conditions
2. Satisfy boundary conditions: compute eigenvalues and eigenfunctions
   - Solve an ODE boundary value problem
   - Compute eigenvalues corresponding to nontrivial (nonzero) solutions
3. Satisfy initial conditions
   - Use superposition to write the general solution to the PDE
   - Compute particular solution satisfying initial data

Step 1: Separate variables

Solve for $u(x,t)$:

$$
\begin{aligned}
u_t &= k\,u_{xx}, & u(x,0) &= f(x),\\
u(0,t) &= 0, & u(L,t) &= 0,
\end{aligned}
$$

Step 2: Boundary conditions (eigenvalues/eigenfunctions)

$$u_t = k\, u_{xx}, \qquad\qquad u(x,0) = f(x),$$
$$u(0,t) = 0, \qquad\qquad u(L,t) = 0,$$

Step 3: General solution and initial conditions

$$
\begin{aligned}
u_t &= k\, u_{xx}, & u(x,0) &= f(x), \\
u(0,t) &= 0, & u(L,t) &= 0,
\end{aligned}
$$

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More examples, I

Example

Solve for $u(x,t)$:

$$u_t = k\,u_{xx}, \qquad u(x,0) = f(x),$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(L,t) = 0.$$

More examples, II

Example

Solve for $u(x,t)$:

$$u_t = k\, u_{xx}, \qquad u(x,0) = f(x),$$

$$u(0,t) = u(L,t), \qquad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t).$$