Linearity and superposition

MATH 3150 Lecture 03

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Haberman 5th edition: Sections 2.1.2-2

PDEs and the heat equation

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The PDE above can be equivalently described by *linear operators*.

This alternative description allows us to leverage *linearity* to analyze PDE solutions.

PDEs as linear operators

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can equivalently be written as

$$L\,[u] = 0, \qquad\qquad L[u] := \left(\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}\right) u = u_t - k u_{xx}.$$

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L above is a (differential) *operator*.

Similar to linear functions, linear operators satisfy linearity conditions.

Definition

An operator L is linear if, given any 2 functions u_1, u_2, and any 2 real scalars c_1, c_2,

$$L[c_1 u_1 + c_2 u_2] = c_1 L[u_1] + c_2 L[u_2].$$

A PDE is linear if it can be written as $L[u] = f$ for some linear operator L.

Linearity

Example

Show that the heat equation operator, $L = \frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}$, is linear.

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Example

Determine if the operator L defined by $L[u] = \frac{\partial}{\partial x}\left(u^2\right)$ is linear or nonlinear.

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Consider a PDE in one spatial dimension.

- The initial condition $u(x, 0) = f(x)$ is homogeneous if $f = 0$.
- The boundary condition $u(0, t) = T_1(t)$ is a homogeneous Dirichlet condition if $T_1(t) = 0$.
- The boundary condition $u_x(0, t) = T_1(t)$ is a homogeneous Neumann condition if $T_1(t) = 0$.

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Linear, homogeneous equations need not have homogeneous boundary conditions.

Superposition

The main utility we will get out of these classifications is superposition.

Theorem (Principle of Superposition)

If u_1 *and* u_2 *are both solutions to a linear and homogeneous PDE, then* $c_1u_1 + c_2u_2$*, for any constants* c_1 *and* c_2*, is also a solution to the PDE.*

The above property applies only addresses the PDE! It does *not* consider initial and/or boundary conditions.

Examples, I

Example

Verify superposition for the following ODE and solutions u_1, u_2:

$$u''(x) + \frac{\pi^2}{L^2} u = 0, \qquad u(0) = 0, \qquad u(L) = 0.$$

$$u_1(x) = 0, \qquad u_2(x) = \sin\left(\frac{\pi x}{L}\right),$$

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Verify superposition for the following PDE and solutions u_1, u_2:

$$u_t = u_{xx}, \qquad u(0,t) = 0, \qquad u(L,t) = 0$$

$$u_1(x,t) = 0, \qquad u_2(x,t) = \exp\left(-(\pi/L)^2 t\right) \sin\left(\frac{\pi x}{L}\right),$$

Examples, II

Example

Analyze superposition for the following PDE and solutions u_1, u_2:

$$u''(x) + \pi^2 u = 0, \qquad u(0) = 0, \qquad u(1) = 1$$

$$u_1(x) = x, \qquad u_2(x) = x + \sin(\pi x),$$