Department of Mathematics, University of Utah

Topics in Numerical Solutions of PDE

MTH6630 – Section 001 – Spring 2019

Project 2

Finite volume schemes: high-order, systems, and multidimensional problems

Due Wednesday, April 24, 2019

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1. Consider the scalar wave equation:

$$u_t + u_x = 0, \qquad -1 < x \le 1,\ t > 0,$$

with periodic boundary conditions $u(-1,t) = u(1,t)$. Write *both* an ENO and WENO finite volume solver using a Lax-Friedrichs flux. Use a third-order ENO solver and a fifth-order WENO solver ($m = 3$). Your experiments should use two different initial data:

$$u(x,0) = u_1(x) = \exp(\sin \pi x),$$

$$u(x,0) = u_2(x) = \begin{cases} \frac{1}{6}\left[4\phi(x) + \phi(x-\delta) + \phi(x+\delta)\right], & -\frac{4}{5} \le x \le -\frac{3}{5}, \\ 1, & -\frac{2}{5} \le x \le -\frac{1}{5}, \\ 1 - \left|10(x-0.1)\right|, & 0 \le x \le \frac{1}{5}, \\ \frac{1}{6}\left[4\tau(x) + \tau(x-\delta) + \tau(x+\delta)\right], & \frac{2}{5} \le x \le \frac{3}{5}, \\ 0, & \text{else} \end{cases}$$

Here, $\delta = 5 \times 10^{-3}$, and

$$\phi(x) = \exp\left(-\frac{\log 2}{36\delta^2}(x+0.7)^2\right), \qquad \tau(x) = \sqrt{\max\left(1 - 100(x-0.5)^2, 0\right)}.$$

The function u_2 should look like Figure 1.5 in the text.
Using $u_1(x)$ as the initial data, what spatial order of convergence do you observe? (For example, in the L^1 norm.) For the u_2 example, compare results from the W/ENO simulations against those from a first-order finite volume scheme using a Lax-Friedrichs flux.

2. This problem asks you to write solvers for the one-dimensional Euler equations of gas dynamics. These equations, in conservation form, are written as

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial x} = 0, \qquad t > 0,$$

where the conserved variable $\boldsymbol{u}(x,t) \in \mathbb{R}^3$ and flux $\boldsymbol{f}$ are functions of the physical gas density $\rho(x,t)$, velocity $u(x,t)$, pressure $p(x,t)$, and energy $E(x,t)$,

$$\boldsymbol{u} = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \qquad \boldsymbol{f} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{bmatrix},$$

where pressure and energy are related by

$$p = (\gamma - 1)\left(E - \frac{1}{2}\rho u^2\right),$$

where $\gamma = 7/5$ is a physical constant. The flux Jacobian for this problem is given by

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{3-\gamma}{2}u^2 & (3-\gamma)u & \gamma - 1 \\ -\frac{\gamma E u}{\rho} + (\gamma - 1)u^3 & \frac{\gamma E}{\rho} - \frac{3(\gamma-1)u^2}{2} & \gamma u \end{bmatrix}$$

This Jacobian can be diagonalized as follows:

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} = \boldsymbol{S}(\boldsymbol{u})\boldsymbol{\Lambda}(\boldsymbol{u})\boldsymbol{S}(\boldsymbol{u})^{-1},$$

$$\boldsymbol{\Lambda} = \begin{bmatrix} u + c & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u - c \end{bmatrix}.$$

We have introduced the local sound speed c, defined as

$$c \coloneqq \sqrt{\frac{\gamma p}{\rho}},$$

The flux Jacobian and its diagaonlization are also given on pages 52-53 of the textbook.
Implement a finite volume solver for this problem using a Harten-Lax-van Leer (HLL) flux. This flux, given left- and right-hand states $\boldsymbol{u}_l$ and $\boldsymbol{u}_r$, respectively, is evaluated as

$$\boldsymbol{F}(\boldsymbol{u}_l, \boldsymbol{u}_r) = \begin{cases} \boldsymbol{f}(\boldsymbol{u}_l), & s_- \geq 0 \\ \frac{s_+ \boldsymbol{f}(\boldsymbol{u}_l) - s_- \boldsymbol{f}(\boldsymbol{u}_r) + s_+ s_- (\boldsymbol{u}_r - \boldsymbol{u}_l)}{s_+ - s_-}, & s_- \leq 0 \leq s_+ \\ \boldsymbol{f}(\boldsymbol{u}_r), & s_+ \leq 0. \end{cases}$$

Here, $s_\pm$, are the extreme wavespeeds of flux jacobians:

$$s_- \coloneqq \min_i \left(\lambda_i\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{u}_l)\right), \lambda_i\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{u}_r)\right)\right), \qquad s_+ \coloneqq \max_i \left(\lambda_i\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{u}_l)\right), \lambda_i\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{u}_r)\right)\right),$$

where $\lambda_i(\boldsymbol{A})$ denotes the ith eigenvalue of $\boldsymbol{A}$.
Test your solver with two popular initial conditions:

a. *Sod's shock tube* — For $x \in [0, 1]$,

$$\rho(x, 0) = \begin{cases} 1, & x < \frac{1}{2} \\ \frac{1}{8}, & x \geq \frac{1}{2} \end{cases} \qquad \rho u(x, 0) = 0, \qquad E(x, 0) = \frac{1}{\gamma - 1}\begin{cases} 1, & x < \frac{1}{2} \\ \frac{1}{10}, & x \geq \frac{1}{2} \end{cases}$$

Use non-periodic boundary conditions that match the $x = 0$ and $x = 1$ values of the initial data, and solve up to terminal time $T = 0.2$. Plot the density ρ, the velocity u, and the pressure p.

b. *Shock entropy wave* — For $x \in [-5, 5]$,

$$\rho = \begin{cases} 3.857143, & x < -4, \\ 1 + 0.2\sin(\pi x), & x \geq -4 \end{cases}$$

$$u = \begin{cases} 2.629369, & x < -4, \\ 0, & x \geq -4 \end{cases}$$

$$p = \begin{cases} 10.33333, & x < -4, \\ 1, & x \geq -4 \end{cases}$$

Again use non-periodic boundary conditions that match the $x = \pm 5$ values of the initial data. Simulate up to time $T = 1.8$ and again plot the density, velocity, and pressure.

3. In this problem we will solve the scalar two-dimensional Kurganov-Petrova-Popov (KPP) problem:

$$\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{f}(u) = 0, \qquad t > 0,\ \ x \in [-2, 2],\ \ y \in \left[-\frac{5}{2}, \frac{3}{2}\right],$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right), \qquad \boldsymbol{f}(u) = (\sin u, \cos u)^T,$$

with initial data

$$u(x, y, 0) = \begin{cases} \frac{14\pi}{4}, & x^2 + y^2 < 1, \\ \frac{\pi}{4}, & \text{else} \end{cases}$$

Use a finite volume scheme using a two-dimensional Cartesian mesh to solve this problem up to time $T = 1$. Note that the finite volume formulation of this scheme on a Cartesian mesh reduces to a scheme with “essentially” one-dimensional operations. Use a first-order Lax-Friedrichs flux, and a 3rd/5th order Lax-Friedrichs W/ENO flux from the first problem to compute numerical solutions to this problem.