Department of Mathematics, University of Utah
Topics in Numerical Solutions of PDE
MTH6630 – Section 001 – Spring 2019

Project 1
Finite volume schemes
Due Monday, April 1, 2019

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1. Compute the entropy solution to the general Riemann problem for Burgers' equation:

$$u_t + f(u)_x = 0$$
$$u(x,0) = \begin{cases} u_l, & x < 0 \\ u_r, & x \geq 0 \end{cases}$$

where $f(u) = u^2/2$.

2. Consider Burgers' equation:

$$u_t + f(u)_x = 0$$
$$u(x,0) = u_0(x),$$

where $f(u) = u^2/2$. With periodic boundary conditions on $x \in [-1, 1]$, implement finite volume schemes for this problem that use the Godunov, Lax-Friedrichs, Local Lax-Friedrichs (Rusanov), and Roe numerical fluxes. These are defined, respectively, as

$$F_G(u,v) = \begin{cases} \max_{v \leq \sigma \leq u} f(\sigma), & u > v \\ \min_{u \leq \sigma \leq v} f(\sigma), & u \leq v \end{cases}, \qquad F_{LF}(u,v) = \frac{1}{2}\left(f(u) + f(v)\right) - \frac{\alpha}{2}\left(v - u\right),$$

$$F_{Rus}(u,v) = \frac{1}{2}\left(f(u) + f(v)\right) - \frac{\beta(u,v)}{2}\left(v - u\right), \qquad F_{Roe}(u,v) = \begin{cases} f(u), & s'(u,v) > 0 \\ f(v), & s'(u,v) \leq 0, \\ f(u), & u = v \end{cases}$$

The parameters α, β, and s' above are defined as

$$\alpha = \max_{\sigma} |f'(\sigma)|, \qquad \beta(u,v) = \max\{|f'(u)|, |f'(v)|\}., \qquad s'(u,v) = \frac{f(u) - f(v)}{u - v}.$$

Consider the initial data

$$u_0(x) = \begin{cases} 1, & |x| \leq 0.5, \\ -1, & \text{else} \end{cases}$$

Compare snapshots of the solution computed from each the the four schemes at time $T = 0.5$ and $T = 1.5$ against the exact solution. What do you observe about the solution computed from each of the schemes? Are they entropy solutions?

3. Repeat the above problem, but now for the flux and initial data,

$$f(u) = u(1-u), \qquad u_0(x) = \begin{cases} 0.75, & |x| < 0.5, \\ 0, & \text{else} \end{cases}$$

This flux corresponds to a PDE model for traffic flow. Are computed solutions from the four schemes entropy solutions?

4. Implement a *nonperiodic* version of the schemes in problem 2 for the boundary conditions

$$u(-1,t) = 1, \qquad u(1,t) = 0.$$

For Burgers' equation with the initial data,

$$f(u) = u^2/2, \qquad u_0(x) = \begin{cases} 1, & x < 0, \\ 0, & x \geq 0 \end{cases}$$

Plot L^1 errors in the computed solution as a function of the number of finite volume cells for each of the 4 schemes. What are the observed convergence rates for each scheme?

Repeat this experiment for the boundary conditions and initial data:

$$u(-1,t) = -1, \qquad u(1,t) = 1, \qquad u_0(x) = \begin{cases} -1, & x \leq 0, \\ 1, & x > 0 \end{cases}$$

To implement nonperiodic schemes, use the boundary condition values as appropriate inputs to numerical flux functions evaluated at boundaries.