

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH
Analysis of Numerical Methods I
MTH6610 – Section 001 – Fall 2019

Lecture notes – Quadrature
Wednesday November 20, 2019

These notes are not a substitute for class attendance. Their main purpose is to provide a lecture overview summarizing the topics covered.

Quadrature is the numerical approximation of definite integrals. The prototypical example is a one-dimensional integral over a compact interval:

$$\int_a^b f(x) dx, \quad -\infty < a < b < \infty.$$

Perhaps the most common type of quadrature uses N point evaluations of f to approximate this integral:

$$\int_a^b f(x) dx \approx \sum_{j=1}^N w_j f(x_j),$$

where the x_j are the quadrature *abscissae* or *nodes* and the w_j are the quadrature *weights*, both of which must be prescribed or determined.

A popular approach to constructing such quadrature rules is to use polynomial interpolation to construct *interpolatory* quadrature rules. I.e., if one prescribes x_1, \dots, x_N as distinct points in $[a, b]$, then the unique degree- $(N - 1)$ polynomial that interpolates $f(x)$ at these points is given by

$$p_{N-1}(x) = \sum_{j=1}^N f(x_j) \ell_j(x),$$

where ℓ_j are the N cardinal Lagrange interpolating polynomials associated with the points x_1, \dots, x_N . An interpolatory quadrature rule constructs a quadrature rule via the approximation

$$\int_a^b f(x) dx \approx \int_a^b p_{N-1}(x) dx = \sum_{j=1}^N f(x_j) \int_a^b \ell_j(x) dx,$$

which shows that we can define the quadrature weights as

$$w_j = \int_a^b \ell_j(x) dx.$$

This provides a formulaic procedure for computing interpolatory quadrature rules once x_1, \dots, x_N are prescribed.

There is a different, equivalent procedure for computing the weights w_j of an interpolatory quadrature rule for the given nodes x_1, \dots, x_N . We note that

$$f \in \text{span} \{1, x, x^2, \dots, x^{N-1}\} \Rightarrow f(x) = p_{N-1}(x),$$

so that an interpolatory quadrature rule with the weights w_j satisfies

$$\int_a^b p(x) dx = \sum_{j=1}^N p(x_j)w_j, \quad p \in \text{span} \{1, x, x^2, \dots, x^{N-1}\}.$$

Noting that the above constraint is linear in the unknowns w_j , then let q_1, \dots, q_N be any basis for $\text{span} \{1, x, \dots, x^{N-1}\}$, so that the vector solution $\mathbf{w} \in \mathbb{R}^N$ to the linear system

$$\mathbf{V}\mathbf{w} = \mathbf{b}, \quad (V)_{k,j} = q_k(x_j), \quad (b)_k = \int_a^b q_k(x) dx,$$

contains the interpolatory quadrature rule weights w_j . Solving the above linear system is an equivalent way of computing weights w_j for an interpolatory rule.

The error committed by an interpolatory quadrature rule can be understood by considering the error committed by the interpolation. The following is a standard error formula committed by polynomial interpolation on an interval $[a, b]$:

$$f(x) - p_{N-1}(x) = \prod_{j=1}^N (x - x_j) \frac{f^{(N)}(\xi)}{N!}, \quad \xi = \xi(x) \in [a, b].$$

The number ξ is usually not computable. However, noting that

$$\left| \prod_{j=1}^N (x - x_j) \right| \leq (b - a)^N,$$

then

$$\max_{x \in [a,b]} |f(x) - p_{N-1}(x)| \leq \frac{(b - a)^N}{N!} \max_{x \in [a,b]} |f^{(N)}(x)|,$$

so that the error committed by the interpolatory quadrature rule is

$$\left| \int_a^b f(x) dx - \int_a^b p_{N-1}(x) dx \right| \leq (b - a) \max_{x \in [a,b]} |f(x) - p_{N-1}(x)| \leq \frac{(b - a)^{N+1}}{N!} \max_{x \in [a,b]} |f^{(N)}(x)|.$$

One frequently uses interpolatory quadrature rules in a *composite* (i.e., piecewise) form, so that the interval length $(b - a)$ is usually small, $(b - a) < 1$. Under this assumption, one then sees that this provides a convergent interpolatory estimate, so long as the N th derivative of f does not become too large. In particular, the error scales as $(b - a)^{N+1}$ as $(b - a) \rightarrow 0$. We say that the *order* of the quadrature rule is $N + 1$.

A special family of interpolatory quadrature rules is given by the Newton-Cotes rules. These are N -point ($N \geq 2$) rules using equidistant points on $[a, b]$. If the nodes include the endpoints, these are called *closed* Newton-Cotes rules. If they do not include the endpoints, they are open Newton-Cotes rules.

Finally, we recall that interpolation on equidistant nodes is usually a bad idea for high-degree polynomial interpolation. Therefore, the Newton-Cotes rules are only viable when $N \lesssim 10$.