

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH
Analysis of Numerical Methods I
MTH6610 – Section 001 – Fall 2019

Lecture notes – Fourier Series
Monday, November 11, 2019

These notes are **not** a substitute for class attendance. Their main purpose is to provide a lecture overview summarizing the topics covered.

We consider the approximation of functions on finite intervals via a Fourier Series. This requires some notation. We consider the interval $[0, 2\pi]$, and introduce the space of square-integrable periodic functions:

$$L^2 = L^2_p([0, 2\pi]; \mathbb{C}) = \{f : [0, 2\pi] \rightarrow \mathbb{C} \mid \|f\|_{L^2} < \infty, \text{ and } f(0) = f(2\pi)\},$$

where

$$\|f\|_{L^2}^2 = \langle f, f \rangle, \quad \langle f, g \rangle = \int_0^{2\pi} f(x)g(x) \, dx$$

We will also need integer Sobolev spaces; for $s = 0, 1, \dots$, define

$$H^s = H^s_p([0, 2\pi]; \mathbb{C}) = \left\{ f \in L^2 \mid f^{(r)} \in L^2, r = 0, \dots, s, \text{ and } f^{(r)}(0) = f^{(r)}(2\pi), r = 0, \dots, s-1 \right\},$$

where $f^{(s)}$ is the s th derivative of f . Note that $L^2 = H^0$. We define the norm on this space as

$$\|f\|_{H^s}^2 = \sum_{k=0}^s \left\| f^{(k)} \right\|_{L^2}^2.$$

Define the L^2 -orthonormal functions

$$v_j(x) = \frac{1}{\sqrt{2\pi}} e^{ijx}, \quad j \in \mathbb{Z},$$

where $i = \sqrt{-1}$ is the imaginary unit. The family $\{v_j\}_{j \in \mathbb{Z}}$ is complete in L^2 . Therefore,

$$f \in L^2 \implies f(x) = \sum_{j \in \mathbb{Z}} \widehat{f}_j v_j(x),$$

where \widehat{f}_j are some constants. Parseval's identity states

$$\|f\|_{L^2}^2 = \sum_{j \in \mathbb{Z}} |\widehat{f}_j|^2.$$

Formally taking inner products on both sides of the expansion of $f(x)$ shows that

$$\widehat{f}_j = \langle f, v_j \rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) e^{ijx} \, dx$$

When using f in computational approximations, we frequently represent it as a truncated version of its infinite series. This has the form

$$f_N(x) = \sum_{|j| < N} \widehat{f}_j v_j(x), \quad N \in \mathbb{N}.$$

A natural question is how large of an error we make in approximating f by f_N . To answer this question, first suppose that $f \in H^s$ for some $s > 0$. We have computed the expansion coefficients \widehat{f}_j of f , but we can likewise compute the expansion coefficients $\widehat{f}_j^{(1)}$ of $f^{(1)}$:

$$f \in H^1 \implies f^{(1)} \in L^2 \implies f^{(1)} = \sum_{j \in \mathbb{Z}} \widehat{f}_j^{(1)} e^{ijx},$$

where

$$\widehat{f}_j^{(1)} = \langle f^{(1)}, v_j \rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f^{(1)} e^{ijx} dx.$$

A usage of integration by parts yields the identity

$$\widehat{f}_j^{(1)} = ij \widehat{f}_j \implies |\widehat{f}_j|^2 = \frac{1}{j^2} |\widehat{f}_j^{(1)}|^2.$$

This identity will be useful in bounding the error committed by replacing f by f_N , since

$$\begin{aligned} \|f - f_N\|_{L^2}^2 &= \sum_{|j| \geq N} |\widehat{f}_j|^2 = \sum_{|j| \geq N} \frac{1}{j^2} |\widehat{f}_j^{(1)}|^2 \leq \frac{1}{N^2} \sum_{|j| \geq N} |\widehat{f}_j^{(1)}|^2 \\ &\leq \frac{1}{N^2} \sum_{j \in \mathbb{Z}} |\widehat{f}_j^{(1)}|^2 \leq \frac{1}{N^2} \|f^{(1)}\|_{L^2}^2 \leq \frac{1}{N^2} \|f\|_{H^1}^2. \end{aligned}$$

Repeating this argument by induction yields the following classical approximation inequality:

$$f \in H^s \implies \|f - f_N\|_{L^2} \leq N^{-s} \|f\|_{H^s}.$$

This statement has a simple interpretation: *smoothness* of f , i.e., existence of sufficiently many derivatives, implies faster convergence when approximating by Fourier Series.