

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH
Analysis of Numerical Methods I
MTH6610 – Section 001 – Fall 2019

Lecture notes: Conditioning
Friday, September 20, 2019

These notes are not a substitute for class attendance. Their main purpose is to provide a lecture overview summarizing the topics covered.

Reading: Trefethen & Bau III, Lectures 12

We have seen that some algorithms are sensitive to small changes in inputs to the problem. However, this sensitivity could stem from one of two things: (1) either the inherent sensitivity of the exact mathematical problem to inputs, or (2) ill-behavior of algorithms that operate in finite-precision arithmetic. Here we consider the codification of the first of these issues, which is called the *conditioning* of a problem.

Consider a general (not necessarily linear) function $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$, with some given norms on \mathbb{C}^n and \mathbb{C}^m . We will denote both of these norms as $\|\cdot\|$. At some given $x \in \mathbb{C}^n$, we are interested in studying the sensitivity of $f(x)$ with respect to small perturbations of the input x . A reasonable measure of this sensitivity is the relative error when we perturb x by δx ,

$$\frac{\|f(x + \delta x) - f(x)\|}{\|\delta x\|}.$$

Making the notational definition $\delta f = f(x + \delta x)$, then the absolute condition number of f at x is the worst relative error when infinitesimal perturbations are made:

$$\widehat{\kappa}(x) = \lim_{\delta \rightarrow 0} \sup_{\|\delta x\| \leq \delta} \frac{\|\delta f\|}{\|\delta x\|}.$$

Note that this definition is a property of the function f itself, and *not* of any algorithm that implements f . Since x and $f(x)$ may actually be quite large in norm themselves, it is frequently more appropriate to work with the relative condition number, which normalizes by these values,

$$\kappa(x) = \lim_{\delta \rightarrow 0} \sup_{\|\delta x\| \leq \delta} \frac{\|\delta f\| \|x\|}{\|f(x)\| \|\delta x\|}$$

If f is a smooth function of its inputs, then

$$\widehat{\kappa}(x) = \|J(x)\|, \quad J(x) = \frac{\partial f}{\partial x}(x)$$

where $J \in \mathbb{C}^{m \times n}$ is the Jacobian matrix of f . Here, $\|J\|$ is the matrix norm induced by the given norms on \mathbb{C}^m and \mathbb{C}^n .

These definitions immediately allow us to consider the conditioning of some familiar linear problems.

- Let $A \in \mathbb{C}^{n \times n}$ be nonsingular and $x \in \mathbb{C}^n$ be given. Evaluating $y = Ax$ for perturbations in x leads us to consider the function $x \mapsto Ax$. This function has relative condition number $\kappa(x) \leq \|A\| \|A^{-1}\|$. Equality is achieved for certain x vectors that depend on A . This result also applies to the problem $b \mapsto A^{-1}b$ for given b .
- Let $A \in \mathbb{C}^{n \times n}$ be nonsingular and $b \in \mathbb{C}^n$ be given. Evaluating $y = Ax$ for perturbations in A leads us to consider the function $A \mapsto A^{-1}b$. This function has relative condition number $\kappa(x) = \|A\| \|A^{-1}\|$.

For some fixed nonsingular matrix A , the quantity $\kappa(A) = \|A\| \|A^{-1}\|$ is called the condition number of the matrix A . When A is rectangular and we use the induced 2-norm $\|\cdot\|$, we can use $\kappa(A) = \|A\| \|A^+\| = \sigma_1(A)/\sigma_r(A)$, where $r = \text{rank}(A)$.