

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH
Analysis of Numerical Methods I
MTH6610 – Section 001 – Fall 2017

Lecture notes
Monday September 9, 2019

These notes are not a substitute for class attendance. Their main purpose is to provide a lecture overview summarizing the topics covered.

Reading: Trefethen & Bau III, Lecture 6

We consider projection matrices. A matrix $P \in \mathbb{C}^{n \times n}$ is a projection matrix if and only if it is *idempotent*:

$$P^2 = P$$

A projection matrix is an *orthogonal projector* if, in addition to being idempotent,

$$P = P^*$$

A projection matrix that is not orthogonal is called *oblique*. If P is a(n orthogonal) projection matrix, then $I - P$ is also a(n orthogonal) projection matrix.

Projection matrices are uniquely determined by their range and their kernel. In particular,

$$P^2 = P \quad \implies \quad \text{range}(P) \oplus \ker(P) = \mathbb{C}^n.$$

In fact, if $\mathcal{V} \subset \mathbb{C}^n$ and $\mathcal{W} \subset \mathbb{C}^n$ satisfy $\mathcal{V} + \mathcal{W} = \mathbb{C}^n$ and $\dim \mathcal{V} + \dim \mathcal{W} = n$ (hence $\mathcal{V} \oplus \mathcal{W} = \mathbb{C}^n$), then there is a unique projection matrix P with range \mathcal{V} and kernel \mathcal{W} . When P is an orthogonal projector, then $\text{range}(P) \perp \ker(P)$.

Given \mathcal{V} and \mathcal{W} satisfying $\mathcal{V} \oplus \mathcal{W} = \mathbb{C}^n$, there are constructive ways to generate the associated unique projector P with range \mathcal{V} and kernel \mathcal{W} . If $\dim \mathcal{V} = r$, let v_1, \dots, v_r be any basis for \mathcal{V} , and let w_1, \dots, w_r be any basis for \mathcal{W}^\perp . Then the projection P is given by

$$P = V(W^*V)^{-1}W^*, \quad V = [v_1, \dots, v_r], \quad W = [w_1, \dots, w_r].$$

Orthogonal projectors in particular can be easily constructed from orthonormal vectors: if q_1, \dots, q_r are any $r \leq n$ orthonormal vectors in \mathbb{C}^n , then the matrix P given by

$$P = QQ^*, \quad Q = [q_1 \ q_2 \ \cdots \ q_r] \in \mathbb{C}^{n \times r},$$

is the orthogonal projector onto the span of the q_j .

All nontrivial projection matrices P satisfy $\|P\|_2 \geq 1$, with equality if and only if P is an orthogonal projector. Orthogonal projection matrices have eigenvalues either 1 or 0.