Department of Mathematics, University of Utah

Analysis of Numerical Methods I

MTH6610 – Section 001 – Fall 2017

Lecture notes

Friday, August 23, 2019

These notes are not a substitute for class attendance. Their main purpose is to provide a lecture overview summarizing the topics covered.

Reading: Trefethen & Bau III, Lectures 4,5

The main goal is to discuss the singular value decomposition (SVD) of $A \in \mathbb{C}^{m \times n}$. Formally, existence of this decomposition is stated as follows:

Theorem 1. *Given $A \in \mathbb{C}^{m \times n}$, there exist matrices U, Σ, and V of sizes $m \times m$, $m \times n$, and $n \times n$, respectively, such that*

$$A = U \Sigma V^*,$$

where U and V are unitary matrices, and Σ is a diagonal matrix with real, non-negative diagonal entries.

With $q \coloneqq \min(m, n)$, the diagonal entries of Σ, $\sigma_1, \ldots \sigma_q$ are usually ordered in desecending order. The SVD has an attractive geometric interpretation in terms of hyperellipses, and codifies action of the map $A : \mathbb{C}^n \to \mathbb{C}^m$.
When $r = \text{rank}(A) < q$, there are "reduced" versions of the SVD where one truncates U, Σ, and V to sizes $m \times r$, $r \times r$, and $r \times n$, respectively.
The SVD has numerous attractive uses:

- The SVD is the "general" way to diagonalize matrices. (This is separate from matrix diagonalization via an eigendecomposition.)
- The matrix rank, range, nullspace, 2-norm, and Frobenius norms can be determined explicitly from the SVD.
- The non-zero squared singular values of A match the non-zero eigenvalues of A^*A and AA^*.
- The SVD provides an explicit strategy for low-rank approximation of matrices.