Department of Mathematics, University of Utah

# Analysis of Numerical Methods I
## MATH 6610 – Section 001 – Fall 2019
## Homework 3
## *LU* and Cholesky factorizations

**Due Monday, November 4, 2019 by by 11:59pm MT**

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**Submission instructions**:
Create a private repository on `github.com` named `math6610-homework-3`. Add your LaTeX source files and your Matlab/Python code and push to Github. To submit: grant me (username `akilnarayan`) write access to your repository.
You may grant me write access before you complete the assignment. I will not look at your submission until the due date+time specified above. If you choose this route, I will only grade the assignment associated with the last commit before the due date.
All commits timestamped after the due date+time will be ignored.

**Problem assignment**:
Trefethen & Bau III, Lecture 20: # 20.1
Trefethen & Bau III, Lecture 21: # 21.6
Trefethen & Bau III, Lecture 22: # 22.3
Trefethen & Bau III, Lecture 23: # 23.1
Trefethen & Bau III, Lecture 24: # 24.1, 24.4

Additional problems:

**P1.** Let $P \in \mathbb{R}^{n\times n}$ be a permutation matrix.

- (a) Prove that $P^T$ is also a permutation matrix.
- (b) Prove that $P^T = P^{-1}$.
- (c) Prove that if $P_1$ and $P_2$ are both permutation matrices, then $P_1 P_2$ is also a permutation matrix.
- (d) Is it true in general that $P^2 = P$? If so, prove it. If not, give a counterexample.

**P2.** Let $A \in \mathbb{C}^{n\times n}$ be invertible. Prove that the $LU$ decomposition algorithm with partial pivoting always successfully computes $PA = LU$.

**P3.** Given $A \in \mathbb{C}^{m\times n}$, consider a *column-pivoted* QR decomposition, i.e., a factorization of the form,

$$AP = QR,$$

where $P$ is a permutation matrix that is chosen in the following way: At step $j$ in the orthogonalization process (say step $j$ of Gram-Schmidt), the columns $j, j+1, \ldots, n$ are permuted/pivoted so that $r_{jj}$ will be as large as possible.

Note that the vector $p$ defined as

$$p := P^T \begin{pmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{pmatrix} \in \mathbb{R}^n$$

has entries that identify the *column pivots*, i.e., the ordered column indices of $A$ chosen by the pivoting process.

(a) (Column-pivoted QR decompositions are *rank-revealing*, in a sense.) Prove that the number of nonzero diagonal entries in $R$ equals the rank of $A$.

(b) (Column-pivoted QR is greedy determinant maximization.) Let $r = \operatorname{rank}(A)$. For $S$ any subset of $\{1, 2, \ldots, n\}$, let $A_S$ denote the $m \times |S|$ submatrix of $A$ formed by selected the column indices in $S$. Furthermore, let $G_S \in \mathbb{C}^{|S| \times |S|}$ be defined as

$$G_S = (A_S)^* A_S$$

Consider the following iterative, greedy, determinant maximization:

$$s_j = \operatorname{argmax}_{k \in \{1,\ldots,n\}} \det G_{S_{j-1} \cup \{k\}}, \qquad S_j := S_{j-1} \cup \{s_j\}.$$

Where $S_0 := \{\}$. If each maximization yields a unique $s_j$, show that $p_j = s_j$ for $j = 1, \ldots, r$.

**P4.** Let $A \in \mathbb{C}^{m \times n}$, and let $r = \operatorname{rank}(A)$. Show that the LU factorization with partial row pivoting,

$$PA = LU,$$

selects pivots via another kind of greedy determinant maximization. I.e., with $S$ as in the previous problem, let ${}_S A$ denote the $|S| \times n$ matrix formed by selecting the rows with indices $S$ from $A$. Consider the optimization problem

$$s_j = \operatorname{argmin}_{k \in \{1,\ldots,m\}} \left| \det \ {}_{S_{j-1} \cup \{k\}} A_{\{1,\ldots,j\}} \right|$$

for $j \geq 1$ where again $S_0 = \{\}$. If each maximization yields a unique $s_j$, show that, for $j = 1, \ldots, r$, $s_j$ is the $j$th entry of the vector $p$ defined by

$$p := P \begin{pmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ m \end{pmatrix} \in \mathbb{R}^m$$