Department of Mathematics, University of Utah

Analysis of Numerical Methods I

MATH 6610 – Section 001 – Fall 2019

Homework 2

Orthogonalization and the QR decomposition

Due Monday, September 30, 2019 by 11:59pm MT

Submission instructions:
Create a private repository on `github.com` named `math6610-homework-2`. Add your LaTeX source files and your Matlab/Python code and push to Github. To submit: grant me (username `akilnarayan`) write access to your repository.
You may grant me write access before you complete the assignment. I will not look at your submission until the due date+time specified above. If you choose this route, I will only grade the assignment associated with the last commit before the due date.
All commits timestamped after the due date+time will be ignored.

Problem assignment:
In all the following exercises, if a program in Matlab is requested, replace those instructions with a request to write a program in the language of your choice.

Trefethen & Bau III, Lecture 6: # 6.2, 6.4
Trefethen & Bau III, Lecture 7: # 7.3, 7.5
Trefethen & Bau III, Lecture 9: # 9.2 (You need not internalize lecture #9 to do this problem, nor must you use or know how to use Matlab)
Trefethen & Bau III, Lecture 10: # 10.1, 10.2
Trefethen & Bau III, Lecture 12: # 12.3

Additional problems:

P1. Let $M \in [1, \infty)$ and $n \geq 2$ be arbitrary. Explicitly construct a projection matrix $P \in \mathbb{R}^{n \times n}$ such that $\|P\|_2 = M$.

P2. Let P be a projection matrix. Show that $\ker(P) \perp \operatorname{range}(P)$ if and only if $P = P^*$. (I.e., I am asking you to prove a result that justifies the definition of an orthogonal projector.)

P3. Let $A \in \mathbb{C}^{m \times n}$ have rank r and *reduced* SVD

$$A = \widetilde{U}\widetilde{\Sigma}\widetilde{V}^*$$

The *Moore-Penrose pseudoinverse* of A is defined as

$$A^+ = \widetilde{V}\widetilde{\Sigma}^{-1}\widetilde{U}^*$$

Prove the following:

(a) If A is invertible (hence also square), then $A^{-1} = A^+$.

(b) Prove that the matrices AA^+, A^+A, $I - AA^+$, and $I - A^+A$ are all projection matrices. Are they orthogonal? How would you characterize their ranges and kernels in terms of the fundamental subspaces defined by the matrix A?

(c) Give and prove conditions on A so that $AA^+A = A$.

(d) Show that $\|A^+\|_2 = 1/\sigma_r(A)$

(e) In the induced matrix ℓ^2 norm $\|\cdot\|$, is the operation $A \mapsto A^+$ well-conditioned? That is, given an arbitrary but fixed A and a perturbation matrix B, is $||(A+B)^+ - A^+||/||A^+||$ controllable by $||B||/||A||$? Prove it, or give a counterexample.

P4. Let $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^n$ be given. Show that $x = A^+b$ solves

$$\min_{z\in\mathbb{C}^n} ||z||_2 \quad \text{subject to} \quad \|Az - b\|_2 \text{ is minimized,}$$

What is the difference between this x and the linear least-squares solution to $Az = b$?