Matrix diagonalization

MATH 2250 Lecture 32
Book section 6.2

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Eigenvalues and eigenvectors

Given a square $n \times n$ matrix $\boldsymbol{A}$, if the equality

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v},$$

holds for some $\boldsymbol{v} \neq \boldsymbol{0}$ and arbitrary scalar λ, then λ is an **eigenvalue** and $\boldsymbol{v}$ is an **eigenvector**.

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holds for some $\boldsymbol{v} \neq \boldsymbol{0}$ and arbitrary scalar λ, then λ is an **eigenvalue** and $\boldsymbol{v}$ is an **eigenvector**. Some recap:

- $\lambda = 0$ is fine, $\boldsymbol{v} = \boldsymbol{0}$ is not.
- Eigenvectors are non-unique; in fact, the (infinite) set of eigenvectors associated to a fixed eigenvalue λ is a subspace, and is usually called an eigenspace.
- Eigenvalues can be computed as the roots of a degree-n polynomial via the characteristic equation.
- There are *exactly* n eigenvalues of $\boldsymbol{A}$, counting multiplicity.
- The dimension of an eigenspace for λ is at most the multiplicity of λ.

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In fact, this is true generally:

Theorem

Suppose $\lambda_1, \ldots, \lambda_k$ are all distinct eigenvalues of $\boldsymbol{A}$, associated to eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$, respectively. Then $\{\boldsymbol{v}_1, \ldots \boldsymbol{v}_k\}$ are linearly independent.

Linear independence of eigenvectors (2/2)

Note: if $\boldsymbol{A}$ has n *d*istinct eigenvalues, then there are n linearly independent eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$.

In this case, we have:

$$\boldsymbol{A}\boldsymbol{v}_1 = \lambda_1 \boldsymbol{v}_1, \;\; \cdots \;\; \boldsymbol{A}\boldsymbol{v}_n = \lambda_1 \boldsymbol{v}_n,$$

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which is equivalent to

$$\boldsymbol{A}\begin{pmatrix} | & | & & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & & \boldsymbol{v}_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ \lambda_1\boldsymbol{v}_1 & \lambda_2\boldsymbol{v}_2 & & \lambda_n\boldsymbol{v}_n \\ | & | & & | \end{pmatrix},$$

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Diagonalization

We have shown that if n linearly indpendent eigenvectors exist, we have

$$\boldsymbol{A}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{\Lambda},$$

where

- $\boldsymbol{V}$ is an $n \times n$ matrix whose columns are linearly independent eigenvectors
- $\boldsymbol{\Lambda}$ is an $n \times n$ *d*iagonal matrix whose entries are eigenvalues of $\boldsymbol{A}$

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We now use some linear algebra: if $\boldsymbol{V}$ is square and has linearly independent columns, then it is invertible, i.e., $\boldsymbol{V}^{-1}$ exists. Therefore, we have

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If, given a matrix $\boldsymbol{A}$, there exists a diagonal matrix $\boldsymbol{\Lambda}$ and an invertible matrix $\boldsymbol{V}$ such that the above is true, we say that $\boldsymbol{A}$ is **diagonalizable**.

Given $\boldsymbol{A}$, we **diagonalize** $\boldsymbol{A}$ by writing it in the form (1).

Examples (1/2)

Example (Example 6.2.1)

Diagonalize $\boldsymbol{A}$, or show that it is not diagonalizable.

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Diagonalize $\boldsymbol{A}$, or show that it is not diagonalizable.

$$\boldsymbol{A} = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$$

Examples (2/2)

Example (Example 6.2.3)

Diagonalize $\boldsymbol{A}$, or show that it is not diagonalizable.

$$\boldsymbol{A} = \begin{pmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{pmatrix}$$

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Example (Example 6.2.4)

Diagonalize $\boldsymbol{A}$, or show that it is not diagonalizable.

$$\boldsymbol{A} = \begin{pmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{pmatrix}$$