More Laplace transform manipulations

MATH 2250 Lecture 29
Book section 10.4

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Laplace transforms

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Computing inverse Laplace transforms is, in general, rather difficult.

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Today, we'll investigate multiplication, differentiation, and integration of transforms.

Differentiation (1/2)

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$$\begin{aligned} F'(s) = \frac{\mathrm{d}}{\mathrm{d}s} F(s) &= \frac{\mathrm{d}}{\mathrm{d}s} \int_0^\infty f(t) e^{-st} \mathrm{d}t \\ &= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s} \left[f(t) e^{-st} \right] \mathrm{d}t \\ &= \int_0^\infty -t f(t) e^{-st} \mathrm{d}t = \mathcal{L}\{-t f(t)\} \end{aligned}$$

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\end{aligned}
$$

Thus:

$$
F(s) = \mathcal{L}\{f(t)\} \qquad \Longrightarrow \qquad F'(s) = \mathcal{L}\{-t f(t)\}.
$$

I.e., "differentiation in s corresponds to multiplication by $-t$".

Differentiation (2/2)

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Example (Example 10.4.7)

Compute the inverse Laplace transform of $F(s) = \frac{2s}{(s^2-1)^2}$.

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Example (Example 10.4.3)

Compute the Laplace transform of $f(t) = t^2 \sin(kt)$.

Multiplication of transforms (1/2)

We now investigate what it means to multiply two Laplace transforms. Let,

$$F(s) = \mathcal{L}\{f(t)\}, \qquad\qquad G(s) = \mathcal{L}\{g(t)\}.$$

What function $h(t)$ has Laplace transform $F(s)G(s)$?

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What function $h(t)$ has Laplace transform $F(s)G(s)$? We explicitly have:

$$\begin{aligned}
F(s)G(s) &= \int_0^\infty f(\tau)e^{-s\tau}\mathrm{d}\tau \int_0^\infty g(r)e^{-sr}\mathrm{d}r \\
&= \int_0^\infty f(\tau)e^{-s\tau} \underbrace{\int_0^\infty g(r)e^{-sr}\mathrm{d}r}_{r=t-\tau}\,\mathrm{d}\tau \\
&= \int_0^\infty f(\tau)e^{-s\tau} \int_\tau^\infty g(t-\tau)e^{-s(t-\tau)}\mathrm{d}t\mathrm{d}\tau \\
&= \int_0^\infty \int_0^\infty e^{-st} f(\tau)g(t-\tau)\mathrm{d}t\mathrm{d}\tau
\end{aligned}$$

Multiplication of transforms (2/2)

$$F(s) = \mathcal{L}\{f(t)\}, \qquad\qquad G(s) = \mathcal{L}\{g(t)\}.$$

$$\begin{aligned}
F(s)G(s) &= \int_0^\infty \int_0^\infty e^{-st} f(\tau) g(t-\tau) \mathrm{d}t \mathrm{d}\tau \\
&= \int_0^\infty \int_0^\infty e^{-st} f(\tau) g(t-\tau) \mathrm{d}\tau \mathrm{d}t \\
&= \int_0^\infty e^{-st} \int_0^\infty f(\tau) g(t-\tau) \mathrm{d}\tau \mathrm{d}t \\
&= \int_0^\infty e^{-st} \underbrace{\int_0^t f(\tau) g(t-\tau) \mathrm{d}\tau}_{h(t)} \, \mathrm{d}t
\end{aligned}$$

Thus, we have

$$\mathcal{L}\{h(t)\} = F(s)G(s), \qquad\qquad h(t) := \int_0^t f(\tau) g(t-\tau) \mathrm{d}\tau.$$

Convolutions

The operation defining $h(t)$ seems very strange, but it arises very often in engineering models.

Given $f(t)$ and $g(t)$, the **convolution** of f and g is defined as

$$\begin{aligned}(f * g)(t) &:= \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau \\ &= \int_0^t g(\tau)f(t-\tau)\mathrm{d}\tau\end{aligned}$$

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And so we have:

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s),$$

so that "multiplication in s is convolution in t".

Examples

Example

Compute the convolution of $f(t) = \sin t$ and $g(t) = \cos t$.

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Example

Write the solution to the DE

$$x'' + 4x = f(t), \qquad\qquad x(0) = x'(0) = 0,$$

as a convolution of f with another function.