Higher order linear equations

MATH 2250 Lecture 21
Book section 5.2

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nth order linear DE's

The last lecture considered several properties of second order linear DE's: superposition, existence and uniqueness, linear independence, Wronskians, characteristic equations.

There is nothing special about second order equations: All of these concepts carry over to general nth order linear DE's.

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A general nth order **linear** DE has the form

$$y^{(n)} + p_1(x)y^{(n-1)} + p_2(x)y^{(n-2)} + \cdots + p_n(x)y = f(x)$$

It is **homogeneous** if $f(x) \equiv 0$, otherwise it is **nonhomogeneous**.

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Linear DE's obey the principle of **superposition**: if $y_1, \ldots y_n$ are all solutions to homogeneous version of the linear DE above,

$$c_1 y_1(x) + \cdots + c_n y_n(x) = \sum_{j=1}^{n} c_j y_j(x),$$

is also a solution to the DE.

Existence, uniqueness, independence

If $p_1, \dots p_n$, and f are all continuous functions on the open interval I containing the point a, then the initial value problem

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = f(x),$$
$$y(a) = b_0, \;\; y'(a) = b_1, \;\; \cdots \;\; y^{(n-1)}(a) = b_{n-1},$$

has a unique solution for any constants $b_0, b_1, \dots, b_{n-1}$.

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A set of functions $\{y_1, \ldots y_n\}$ are **linearly independent** on the interval I if the condition

$$\sum_{j=1}^{n} c_j y_j(x) = 0, \quad \text{for all } x \text{ in } I$$

is true only when $c_1 = c_2 = \cdots = c_n = 0$. Otherwise they are **lienarly dependent**.

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Determine whether $f_1(x) = \exp(x)$, $f_2(x) = \cosh(x)$, and $f_3(x) = \exp(-x)$ are linearly independent or dependent.

Wronskians

Linear independence of solutions ensures that we can solve initial value problems uniquely.
A linear algebraic way of characterizing this is with Wronskians.

Given n functions $f_1, f_2, \ldots, f_n$, their **Wronskian** determinant is the function

$$W(f_1, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

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Example (Problem 5.2.9)

Use the Wronskian to show that the following functions are linearly independent on $\mathbb{R}$:

$$f_1(x) = \exp(x), \qquad f_2(x) = \sin(x), \qquad f_3(x) = \cos(x)$$

These functions are solutions to the DE $y''' - y'' + y' - y = 0$. Compute the particular solution satisfying the initial data $y(0) = 1$, $y'(0) = 2$, $y''(0) = 0$.

Wronskians and general solutions

Wronskians are a test for when we have a general solution.

Assume that $p_1, \ldots, p_n$ are all continuous functions on an open interval I, and consider the DE

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0,$$

Let $y_1, \ldots, y_n$ be solutions to this DE. Tnen

- $y_1, \ldots, y_n$ are linearly dependent if and only if $W(y_1, \ldots, y_n)(x) = 0$ for each x in I.
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In the latter case, <u>any</u> solution $Y(x)$ to the DE has the form

$$Y(x) = \sum_{j=1}^{n} c_j y_j(x),$$

and hence Y represents the **general solution**.
The solution space for this DE is a vector space, $\mathrm{span}\{y_1, \ldots, y_n\}$ with basis $\{y_1, \ldots y_n\}$ and has dimension n.

Nonhomogenous equations

Linearity allows us to solve nonhomogeneous equations if we know one solution: consider a general solution $Y(x)$ to the DE

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = f(x).$$

Suppose we know any **particular** solution $y_p(x)$ to the DE above.

I.e., both Y and y_p satisfy the DE.

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Suppose we know any **particular** solution $y_p(x)$ to the DE above.

I.e., both Y and y_p satisfy the DE.

Then the function $y_c(x) := Y(x) - y_p(x)$ satisfies the DE

$$y_c^{(n)} + p_1(x)y_c^{(n-1)} + \cdots + p_n(x)y_c = 0,$$

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This identification yields the following result: the general solution $Y(x)$ to the nonhomogeneous DE can be written as

$$Y(x) = y_p(x) + y_c(x),$$

where y_p is any solution, and y_c is the general solution to the associated homogeneous DE.

Nonhomogenous equations

Example (Example 5.2.7)

Verify that $y_p(x) = 3x$ is a solution to

$$y'' + 4y = 12x,$$

and that $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$ is the general solution to the associated homogeneous DE. Find a solution that satisfies $y(0) = 5$, $y'(0) = 7$.