Second order linear equations

MATH 2250 Lecture 20
Book section 5.1

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A *second order* DE has the form

$$G(x, y, y', y'') = 0.$$

This equation is **linear** if the dependence of G on y, y', and y'' is linear.

Second order linear equations

The following are second order linear DE's:

$$
\begin{aligned}
2y'' + y - exp(x) &= 0 \\
y'x^2 &= y'' \sin x \\
A(x)y'' + B(x)y' + C(x)y &= 0
\end{aligned}
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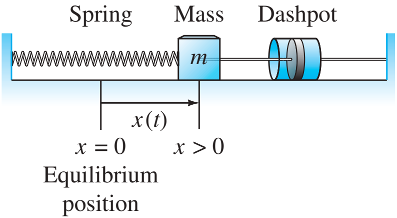
The following are nonlinear DE's:

$$y'' + y^2 = 0$$
$$yy'' + x = 0$$
$$y'' + \sin(y) = 3$$

Mass-spring-damper systems

The mass-spring-damper system is a canonical problem in this course.

Example



The horizontal motion of a block with mass m is governed by a restorative spring force (spring constant k) and a motion damper (damping coefficient c). The differential equation describing the position $x(t)$ of the block is

$$mx'' + cx' + kx = F(t),$$

where $F(t)$ accounts for any external forces acting on the block.

Homogeneous equations

A general second order DE has the form

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where A, B, C, and F are arbitrary functions ($A \not\equiv 0$).

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While the forms of A, B, and C are typically dependent on physics from which the DE is derived, F is typically an external forcing or agent in the system.

It is convenient to assume that $A(x) \neq 0$ for every x in some interval I. Then an equivalent form for a general second order linear DE is

$$y''(x) + p(x)y' + q(x)y = f(x).$$

Superposition

There is a very useful property that linearity for homogeneous equations yields: consider

$$y'' + p(x)y' + q(x)y = 0.$$

If $y_1(x)$ and $y_2(x)$ are both functions that solve the homogeneous DE above, then

$$y(x) = c_1 y_1(x) + c_2 y_2(x),$$

also solves the DE, for **any** constants c_1 and c_2.

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Example

Verify that $y_1(x) = e^x$ and $y_2(x) = e^{-x}$ both solve the DE

$$y'' - y = 0.$$

Show also that $y(x) = 3e^x + 4e^{-x}$ also solves this DE.

Existence and uniqueness

The *linear* property of DE's yields the very important properties of existence and uniqueness:

Suppose that $p(x)$, $q(x)$, and $f(x)$ are all continuous functions on some open interval I that contains the point $x = a$. Then for *any* constants b_0 and b_1, the initial value problem

$$y'' + p(x)y' + q(x)y = f(x), \qquad y(a) = b_0, \qquad y'(a) = b_1,$$

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Verify that $y_1(x) = \cos x$ and $y_2(x) = \sin(x)$ are both solutions to the DE

$$y'' + y = 0$$

Use superposition to compute the unique solution to the DE above paired with the initial data $y(0) = 3$, $y'(0) = -4$.

A little linear algebra

We will consider vector spaces of *functions*. I.e., each point in a vector space is a function.

This means that the "zero vector" in this context is the zero function $y(x) = 0$.

In this context, two functions (i.e., vectors!) $y_1(x)$ and $y_2(x)$ are **linearly independent** if and only if the equation

$$c_1 y_1(x) + c_2 y_2(x) = 0, \qquad \text{for all } x \text{ in } I,$$

is true only when $c_1 = c_2 = 0$. (Otherwise, they are **linearly dependent**.)

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Example

Determine whether or not $f_1(x) = \exp(x)$ and $f_2(x) = \exp(-x)$ are linearly indpendent on $I = \mathbb{R}$.
What about $g_1(x) = \sin(2x)$ and $g_2(x) = \sin x \cos x$?

Wronskians

Consider the initial value problem with the homogeneous DE

$$y'' + p(x)y' + q(x)y = 0, \qquad y(a) = b_0, \qquad y'(a) = b_1.$$

Suppose that $y_1(x)$ and $y_2(x)$ are some solutions to the DE. Then by superposition we can compute a solution if the linear system for (c_1, c_2)

$$\begin{aligned} c_1 y_1(a) + c_2 y_2(a) &= b_0 \\ c_1 y_1'(a) + c_2 y_2'(a) &= b_1 \end{aligned}$$

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Recall: This happens if and only if the we have the following condition on the determinant of the associated matrix:

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$$\begin{vmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{vmatrix} \neq 0$$

This determinant, as a function of x is called the **Wronskian** W:

$$W(f, g) := \det \begin{pmatrix} f & g \\ f' & g' \end{pmatrix} = f(x)g'(x) - f'(x)g(x).$$

The Wronskian and dependence

Note that the Wronskian (determinant) $W(f,g)$ vanishes (equals the zero function) only when f and g are linearly dependent.

This results in a rather strong statement about solutions to DE's and the Wronskian:

The Wronskian and dependence

Note that the Wronskian (determinant) $W(f, g)$ vanishes (equals the zero function) only when f and g are linearly dependent.

This results in a rather strong statement about solutions to DE's and the Wronskian:

Consider two solutions $y_1(x)$ and $y_2(x)$ to the homogeneous DE

$$y'' + p(x)y' + q(x)y = 0,$$

where p and q are continuous on an open interval I.

- y_1 and y_2 are linearly dependent if and only if $W(y_1, y_2)(x) = 0$ for all x in I.
- If y_1 and y_2 are linearly independent if and only if $W(y_1, y_2)(x) \neq 0$ for all x in I.

In the latter case, **any** solution $Y(x)$ to the DE has the form

$$Y(x) = c_1 y_1(x) + c_2 y_2(x)$$

for some constants c_1 and c_2, and so $Y(x)$ is called the **general solution** to the DE.

Constant coefficients DE's

If we specialize to a certain subclass of second order linear DE's, we can compute explicit solutions.

A second order linear DE of the form

$$ay'' + by' + cy = 0,$$

for some constants a, b, and c, is a **constant coefficient** DE. (Also: second order, linear, homogeneous.)

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We compute solutions to this equation by inspection: if we use the *ansatz*

$$y(x) = \exp(rx),$$

for an unknown constant r, then we find that this is a solution only when

$$ar^2 + br + c = 0.$$

The characteristic equation

$$ay'' + by' + cy = 0,$$
$$ar^2 + br + c = 0.$$

This quadratic equation for r is called the **characteristic equation** for the DE.

If the two roots of this equation are **real-valued and distinct**, say $r = r_1$ and $r = r_2$, then

$$y_1(x) = \exp(r_1 x), \qquad y_2(x) = \exp(r_2 x),$$

are a pair of linearly independent solutions.

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are a pair of linearly independent solutions.

Thus, this identifies the general solution, and the **solution space** for this DE is $\mathrm{span}\{y_1, y_2\}$ with basis $\{y_1, y_2\}$.

Examples

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Compute general solutions to the DE's:

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If the roots of the characteristic equation are **real-valued and repeated**, say $r_1 = r_2$, then of course $y_1(x) = \exp(r_1 x)$ is one solution. By inspection, we have that

$$y_2(x) = x \exp(r_1 x)$$

is another solution, and is clearly linearly independent from y_1.

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Example

Compute general solution to the DE

$$y'' + 2y' + y = 0$$