The vector space $\mathbb{R}^3$

MATH 2250 Lecture 16
Book section 4.1

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Vectors in $\mathbb{R}^3$

Every point in $\mathbb{R}^3$ is represented by a 3×1 **vector** containing the point's coordinates.

Vector representations have *both* a magnitude and a direction. They are not just locations.

In this section, we will write all vectors as *row* vectors. This is just for notational simplicity. (Nothing changes if we use column vectors instead.)

Vectors are typically drawn as arrows: the tail is located at the origin and the head is located at the point.

Vectors in $\mathbb{R}^3$

Many vector operations act just like elementwise matrix operations: if $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors and c is a scalar, then

$$c\boldsymbol{u} = \begin{pmatrix} cu_1 & cu_2 & cu_3 \end{pmatrix}, \quad \boldsymbol{u} + \boldsymbol{v} = \begin{pmatrix} u_1 + v_1 & u_2 + v_2 & u_3 + v_3 \end{pmatrix}$$

One more property is that of length:

$$|\boldsymbol{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

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The first two properties are simple, but important: if a collection of objects V has notions of addition and scalar multipliction above that result in outputs that are also in V, then V is called a **vector space**.

$\mathbb{R}^3$ is a vector space

The set of points $\mathbb{R}^3$ is a vector space:

If $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are vectors in $\mathbb{R}^3$ and r and s are any scalars, then:

- $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
- $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$
- $\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{u} = \boldsymbol{u}$ ($\boldsymbol{0}$ a vector with all entries 0)
- $\boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{0}$
- $r(\boldsymbol{u} + \boldsymbol{v}) = r\boldsymbol{u} + r\boldsymbol{v}$
- $(r + s)\boldsymbol{u} = r\boldsymbol{u} + s\boldsymbol{u}$
- $r(s\boldsymbol{u}) = (rs)\boldsymbol{u}$
- $1\,\boldsymbol{u} = \boldsymbol{u}$

The properties above are also practically and mathematically important.
(All of this applies to $\mathbb{R}^2$ as well.)

Linear dependence

Vector spaces afford us a *structure* of the space that we can exploit. For example:

Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **linearly dependent** if there are scalars a and b, that are both **not** zero such that

$$a\boldsymbol{u} + b\boldsymbol{v} = \boldsymbol{0}$$

Vectors that are not linearly dependent are called **linearly independent**.

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Example

For every pair in the set of 4 vectors below, determine if each pair is linearly independent or dependent.

$$(3, -2), \quad (-6, 4), \quad (5, -7), \quad (0, 0).$$

Linear dependence

Note that most linear dependence computations are actually just an exercise in linear systems.

Example

Determine if $\boldsymbol{u} = (3, -2)$ and $\boldsymbol{v} = (-2, 7)$ are linearly independent. If they are, find numbers a and b such that $a\boldsymbol{u} + b\boldsymbol{v} = (11, -4)$.

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The operation above, writing one vector as a sum of others, is most useful when the summands are linearly independent.

Linear dependence for 3 vectors

We have defined linear dependence in such a way that generalizing this concept to 3 (or more) vectors can be done:

Three vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are **linearly dependent** if there are three scalars a, b, and c, which are not all zero, such that

$$a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w} = \mathbf{0}.$$

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We are mostly concerned with situations when independence turns out to be "good" and dependence is "bad".

All these vectors can either be in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Linear dependence via determinants

Testing for linear independence of 3 (or more) vectors is done in the same way as for two vectors: by solving linear systems.

Theorem

Three vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ in $\mathbb{R}^3$ are linearly independent if and only if

$$\det\begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix} \neq 0$$

A basis for $\mathbb{R}^3$

We will be concerned with a **basis** for vector spaces. These are basically a "minimal" collection of vectors that "represent" a vector space.

To understand examples first:

Vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ form a **basis** for $\mathbb{R}^3$ if *any* other vector $\boldsymbol{x}$ in $\mathbb{R}^3$ can be written as

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Equivalently: these vectors are a basis for $\mathbb{R}^3$ if $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are linearly independent.

Subspaces

A final important concept is that of *subspaces*.

Given a vector space V (like $\mathbb{R}^3$), a **subspace** is any subset of V satisfying the definition of a vector space.

I.e., is a subset such that (a) adding two elements in the subspace together results in something in the same subspace and (b) multiplying any element by a scalar results in a vector again in the subspace.

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Subspaces in $\mathbb{R}^3$ are lines and planes passing through the origin.