Equilibrium solutions and stability

MATH 2250 Lecture 07
Book section 2.2

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Long-time behavior

In many DE models, the *long-time* behavior of a solution is physically interesting.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t,x), \qquad\qquad x(t_0) = x_0,$$

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We are interesting in the value of $\lim_{t \to \infty} x(t)$, when such a limit exists.

Intuitively, we expect that the value of this limit depends on the starting location x_0.

To quantitatively understand this behavior, we specialize to a particular class of DE's.

Autonomous equations

A first-order DE for $x(t)$ of the form

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is called an **autonomous** DE.

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Example

The following first-order DE's are autonomous:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x, \qquad x' = sin(x), \qquad x' = x^x - 4.$$

The following first-order DE's are *not* autonomous:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = t, \qquad x' = x + t, \qquad x' = \sin t$$

Equilibrium solutions

Consider an autonomous equation $x' = f(x)$. If $x'(t) = 0$, then the value of x does not change.

I.e., if we find a value x^* such that $f(x^*) = 0$, then the constant solution $x(t) = x^*$ solves the DE.

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Example

Compute the equilibrium solutions for the following DE's:

$$x' = 3 - x, \qquad x' = ax - bx^2, \qquad x' = \sin x,$$

where a and b are positive numbers.

Stability

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Example

Draw phase diagrams for the DE's below, and use this to determine the stability of each equilibrium solution.

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where a and b are positive numbers.

Bifurcations

Autonomous DE's of the form $x' = f(x)$ often involve *parameters*, which are constants in f, e.g.,:

$$x' = x^2 + c,$$

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A plot of the equilibrium solutions versus the parameter (above, c) is called a **bifurcation diagram**.

The values of c at which the qualitative nature of the equilibrium solutions changes is a **bifurcation point**.

Bifurcation examples

Example

Draw the bifurcation diagram and find all bifurcation points for the DE

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$$x' = c(x - x^2)$$