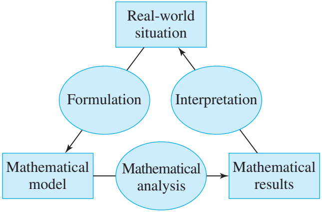
Models and differential equations

MATH 2250 Lecture 01
Book section 1.1

August 20, 2019

Mathematical models



Mathematical variables correspond to values of scalars/fields

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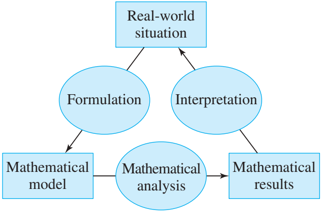
The goal is to predict reality.

Closed-loop modeling cycles use measurements from reality to inform models.

Mathematical models



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All models are wrong, but some are useful.

– George Box

"Science and statistics", Journal of the American Statistical Association, 71: 791–799, doi: 10.1080/01621459.1976.10480949

Differential equations

Differential equations (DE's) are a type of model:
They are equations involving derivatives (of the mathematical variables).

Algebraic equations, e.g., $x^2 + 4x + 4 = 0$ are fundmentally different from DE's.

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The goals of DE modeling are to

- Discover/determine DE models
- *Solve* DE's for variables
- Interpret and understand results

Examples

- Newton's second law of motion
- Newton's law of cooling
- Population growth

Solving differential equations

With *algebraic* equations, we are usually interested in solving for specific values or numbers.

$$x^2 + 4x + 4 = 0, \qquad \log_6(3^{x^2} - 3) = 1$$

Solving differential equations requires *much* more: we require an entire function.

$$y' = 3y \text{ is solved with } y(x) = \exp(3x)$$
$$y(x) = C\exp(x^2) \text{ yields the DE } y' = 2xy$$

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Verifying that a given function satisfies a DE is usually easy (just substitution).
Determining the solution to a given DE is usually quite hard.

Initial data

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A DE coupled with initial conditions is an **initial value problem**.
Generally, DE's *always* require initial data to yield *unique* solutions.

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There are **ordinary differential equations**, involving only ordinary deriatives, and **partial differential equations**, involving partial deriatives.