16.4.10 Solve $u_{tt} = c^2 u_{xx}$, $-\infty < x < \infty$, $t > 0$

$$u(x,0) = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = 0.$$

Define $U(\omega,t) = \mathcal{F}\{u(x,t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)\, e^{i\omega x}\, dx$

Then: $U(\omega,0) = \mathcal{F}\{u(x,0)\} = \mathcal{F}\{f(x)\} = F(\omega)$

$U_t(\omega,0) = \mathcal{F}\{u_t(x,0)\} = 0$

Taking Fourier Transform of PDE: $u_{tt} = c^2 u_{xx}$

$$\downarrow \mathcal{F}$$

$$U_{tt} = c^2(-i\omega)^2 U = -c^2\omega^2 U$$

$$U_{tt} + c^2\omega^2 U = 0$$

Ansatz: $U(\omega,t) = \exp(rt) \Longrightarrow r^2 + c^2\omega^2 = 0$

$r = \pm ic\omega$

$$U(\omega,t) = C_1(\omega)\cos(c\omega t) + C_2(\omega)\sin(c\omega t)$$

$U(\omega,0) = F(\omega) \Longrightarrow C_1(\omega) = F(\omega)$

$U_t(\omega,0) = 0 \Longrightarrow C_2(\omega) = 0$ (Since $U_t(\omega,t) = -c\omega C_1(\omega)\sin(c\omega t) + c\omega C_2(\omega)\cos(c\omega t)$)

$$U(\omega,t) = F(\omega)\cos(c\omega t)$$

So: $u(x,t) = \mathcal{F}^{-1}\{U(\omega,t)\} = \int_{-\infty}^{\infty} F(\omega)\cos(c\omega t)\, e^{-i\omega x}\, d\omega$

Use: $\cos(c\omega t) = \frac{1}{2}\left(\exp(ic\omega t) + \exp(-ic\omega t)\right)$

So: $u(x,t) = \frac{1}{2}\int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega(x-ct)}\, d\omega + \frac{1}{2}\int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega(x+ct)}\, d\omega$

$= \frac{1}{2} f(x-ct) + \frac{1}{2} f(x+ct)$

either by inverse transform definition, or Fourier shift theorem.

P1. (a) $f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx$$

$$= \frac{1}{2\pi}\int_{-1}^{1} 1 \cdot e^{i\omega x}\, dx = \frac{1}{2\pi i\omega}\, e^{i\omega x}\Big|_{x=-1}^{1}$$

$$= \frac{1}{2\pi i\omega}\left[e^{i\omega} - e^{-i\omega}\right] = \frac{\sin\omega}{\pi\omega} \quad \left(\text{since } \sin\theta = \frac{1}{2i}\left(e^{i\theta} + e^{-i\theta}\right)\right)$$

(b) $f(x) = \begin{cases} \exp(-x), & x \ge 0 \\ 0, & x < 0 \end{cases}$

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx = \frac{1}{2\pi}\int_{0}^{\infty} e^{-x+i\omega x}\, dx$$

$$= \frac{1}{2\pi(i\omega-1)}\, e^{-x+i\omega x}\Big|_{x=0}^{x=\infty} = \frac{1}{2\pi(i\omega-1)}\left[\underbrace{\lim_{x\to\infty} e^{-x}}_{=0}\lim_{x\to\infty} e^{i\omega x} - e^{0}\right]$$

$$= \frac{-1}{2\pi(i\omega-1)} = \frac{1}{2\pi(1+i\omega)}$$

(c) $f(x) = \begin{cases} x\exp(-x), & x \geq 0 \\ 0, & x < 0 \end{cases}$

Very similar to (b), except multiplication by x.
Use differentiation property:

$$\mathcal{F}\{ixf(x)\} = \frac{dF}{dw} \Longrightarrow \mathcal{F}\{xf(x)\} = \frac{1}{i}\frac{dF}{dw}$$

Let $f_0(x) = \begin{cases} \exp(-x), & x \geq 0 \\ 0, & x < 0 \end{cases}$

From (b): $F_0(\omega) = \mathcal{F}\{f_0(x)\} = \dfrac{1}{2\pi(1+i\omega)}$

Here, $f(x) = xf_0(x) \Longrightarrow F(\omega) = \dfrac{1}{i}\dfrac{dF_0(\omega)}{dw}$

$$= \frac{1}{i}\frac{d}{dw}\left[\frac{1}{2\pi(1+iw)}\right]$$

$$= \frac{-1}{2\pi i}\frac{1}{(1+iw)^2}\cdot i = \frac{-1}{2\pi(1+iw)^2}$$

(d) $f(x) = \begin{cases} x\cos x\exp(-x), & x \geq 0 \\ 0, & x < 0 \end{cases}$

Defining $f_1(x) = \begin{cases} x\exp(-x), & x \geq 0 \\ 0, & x < 0, \end{cases}$

Part (c) $\Longrightarrow F_1(\omega) = \mathcal{F}\{f_1\} = \dfrac{-1}{2\pi(1+iw)^2}$

And here: $f(x) = \cos x\, f_1(x)$

$$= \frac{1}{2} e^{ix} f_1(x) + \frac{1}{2} e^{-ix} f_1(x)$$

Shift Theorem: $\mathcal{F}\{e^{ix} f_1(x)\} = F_1(\omega + 1)$

So: $\mathcal{F}\{f\} = \frac{1}{2} F_1(\omega+1) + \frac{1}{2} F_1(\omega - 1)$

$$F(\omega) = \frac{-1}{4\pi(1+i+i\omega)^2} + \frac{-1}{4\pi(1-i+i\omega)^2}$$

P2 Solve: $u_t = c u_x - a u$, $-\infty < x < \infty$, $t > 0$

$u(x,0) = f(x)$, $-\infty < c < \infty$, $a > 0$

Since Fourier Transform is linear: $\mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} = \frac{\partial}{\partial t} \mathcal{F}\{u\}$

$$= \frac{\partial U}{\partial t}$$

With $U = \mathcal{F}\{u\}$, taking transform of PDE:

$$U_t = c(-i\omega) U - a U$$
$$= (-a - i\omega c) U$$

$$U(\omega, t) = U(\omega, 0) \exp((-a - i\omega c)t)$$

initial data: $\cancel{u(x)}$ $u(x,0) = f(x) \longrightarrow U(\omega, 0) = \mathcal{F}\{f\} = F(\omega)$

$$U(\omega, t) = F(\omega) \exp((-a - i\omega c)t) = \exp(-at)\, F(\omega) \exp(-i\omega c t)$$

$$u(x,t) = \mathcal{F}^{-1}\{U(\omega, t)\} = \exp(-at)\, \mathcal{F}^{-1}\{F(\omega) \exp(-i\omega c t)\}$$

$$\boxed{u(x,t) = f(x+ct)\, e^{-at}}$$

P3 Solve $u_{tt} = c^2 u_{xx} - a^2 u - 2a u_t, \quad -\infty < x < \infty, \; t > 0$ (5)

$u(x,0) = f(x) \qquad c>0, \; a>0$

$u_t(x,0) = -a f(x)$

Define $U(w,t) = \mathcal{F}\{u(x,t)\}$

$\Longrightarrow \mathcal{F}\{u_{tt}\} = U_{tt}$

$U(w,0) = \mathcal{F}\{u(x,0)\} = F(w) \quad (F(w) = \mathcal{F}\{f\})$

$U_t(w,0) = \mathcal{F}\{u_t(x,0)\} = -aF(w)$

Taking transform of PDE:

$U_{tt} = c^2(-iw)^2 U - a^2 U - 2aU_t$

$U_{tt} + 2aU_t + (a^2 + c^2w^2)U = 0$

characteristic equation: $r^2 + 2ar + (a^2 + c^2w^2) = 0$

$r = \frac{1}{2}\left[-2a \pm \sqrt{4a^2 - 4(a^2+c^2w^2)}\right]$

$= \frac{1}{2}\left[-2a \pm \sqrt{-4c^2w^2}\right]$

$= -a \pm icw$

$U(w,t) = C_1(w)\exp(-at + iwct) + C_2(w)\exp(-at - iwct)$

$U(w,0) = C_1(w) + C_2(w)$

initial data: $U(w,0) = F(w) \Longrightarrow C_1(w) + C_2(w) = F(w)$

$U_t(w,t) = (-a+iwc)C_1(w)\exp(-at+iwct) + (-a-iwc)C_2(w)\exp(-at-iwct)$

$U_t(w,0) = (-a+iwc)C_1(w) - (a-iwc)C_2(w)$

initial data: $U_t(w,0) = -aF(w)$

initial data equations: $C_1(w) + C_2(w) = F(w)$

$(-a+iwc)C_1(w) + (a-iwc)C_2(w) = -aF(w)$

$$-aC_1(\omega) - aC_2(\omega) + i\omega c(C_1(\omega) - C_2(\omega)) = -aF(\omega)$$

$$-a(F(\omega)) + i\omega c(C_1(\omega) - C_2(\omega)) = -aF(\omega)$$

$$i\omega c(C_1(\omega) - C_2(\omega)) = 0$$

$$\Longrightarrow C_1(\omega) = C_2(\omega)$$

$$\Longrightarrow C_1(\omega) = C_2(\omega) = \tfrac{1}{2}F(\omega)$$

$$U(\omega,t) = \tfrac{1}{2}\exp(-at)F(\omega)\exp(i\omega ct) + \tfrac{1}{2}\exp(-at)F(\omega)\exp(-i\omega ct)$$

$$u(x,t) = \tfrac{1}{2}\exp(-at)\mathcal{F}^{-1}\{F(\omega)\exp(i\omega ct)\} + \tfrac{1}{2}\exp(-at)\mathcal{F}^{-1}\{F(\omega)\exp(-i\omega ct)\}$$

$$\boxed{u(x,t) = \tfrac{1}{2}\exp(-at)f(x-ct) + \tfrac{1}{2}\exp(-at)f(x+ct)}$$

P4. Solve:

$$u_t = ku_{xx}, \quad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = \delta(x), \quad k > 0.$$

Define $U(\omega,t) = \mathcal{F}\{u(x,t)\}$

$$\Rightarrow \mathcal{F}\{u_t\} = U_t$$

$$U(\omega,0) = \mathcal{F}\{u(x,0)\} = \mathcal{F}\{\delta(x)\} = \frac{1}{2\pi}$$

Taking transform of PDE:

$$U_t = k(-i\omega)^2 U = -k\omega^2 U$$

$$U(\omega,t) = U(\omega,0)\cdot\exp(-k\omega^2 t)$$

$$= \frac{1}{2\pi}\exp(-k\omega^2 t)$$

$$u(x,t) = \frac{1}{2\pi}\mathcal{F}^{-1}\{\exp(-kt\omega^2)\} = \frac{1}{2\pi}\sqrt{\frac{\pi}{kt}}\exp(-x^2/4kt) = \frac{1}{2\sqrt{\pi kt}}\exp(-x^2/4kt)$$

P5 Solve $u_{tt} = u_{xxxx}$, $-\infty < x < \infty$, $t > 0$

$u(x,0) = f(x)$

$u_t(x,0) = f''(x)$

Define $U(w,t) = \mathcal{F}\{u(x,t)\}$

$\Longrightarrow \mathcal{F}\{u_{tt}\} = U_{tt}$

$U(w,0) = \mathcal{F}\{u(x,0)\} = F(w)$ $\quad (F(w) = \mathcal{F}\{f\})$

$U_t(w,0) = \mathcal{F}\{u_t(x,0)\} = (-iw)^2 F(w)$

$= -w^2 F(w)$

Taking transform of PDE:

$U_{tt} = (-iw)^4 U$

$= w^4 U$

$U_{tt} - w^4 U = 0$

characteristic equation: $r^2 - w^4 = 0$

$r = +w^2, -w^2$

$U(w,t) = C_1(w)\exp(-w^2 t) + C_2(w)\exp(w^2 t)$

$U(w,0) = C_1(w) + C_2(w)$

initial data: $U(w,0) = F(w) \Longrightarrow C_1(w) + C_2(w) = F(w)$

$U_t(w,t) = -w^2 C_1(w)\exp(-w^2 t) + w^2 C_2(w)\exp(w^2 t)$

$U_t(w,0) = -w^2 C_1(w) + w^2 C_2(w)$

initial data: $U_t(w,0) = -w^2 F(w) \Longrightarrow -w^2 C_1(w) + w^2 C_2(w) = -w^2 F(w)$

initial data equations: $C_1(\omega) + C_2(\omega) = F(\omega)$

$$-\omega^2 C_1(\omega) + \omega^2 C_2(\omega) = -\omega^2 F(\omega)$$

$$\left.\begin{array}{l} C_1(\omega) + C_2(\omega) = F(\omega) \\ C_1(\omega) - C_2(\omega) = F(\omega) \end{array}\right\} \Rightarrow \begin{array}{l} C_1(\omega) = F(\omega) \\ C_2(\omega) = 0. \end{array}$$

$$U(\omega, t) = F(\omega)\exp(-\omega^2 t)$$

$$u(x,t) = \mathcal{F}^{-1}\{F(\omega)\exp(-\omega^2 t)\}$$

$$= (f * h)(x,t), \quad \text{where } h = \mathcal{F}^{-1}\{\exp(-\omega^2 t)\}$$

$$= \sqrt{\frac{\pi}{t}}\exp(-x^2/4t)$$

So $u(x,t) = (f*h)(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\, h(x-s)\, ds$

$$= \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(s)\exp(-(x-s)^2/4t)\, ds$$