

MATH 3150 HW #9 Solutions

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Section 16.4 # 3, 4, 6, 7(a, b, c)

16.4.3 (a) $u_t = k u_{xx} + c u_x$, $-\infty < x < \infty$
 $u(x, 0) = f(x)$

$$U(\omega, t) = \mathcal{L}[u(x, t)]$$

$$U_t = k \cdot (-i\omega)^2 U + c(-i\omega)U$$

$$= (-i\omega c - k\omega^2)U$$

$$U(\omega, t) = A(\omega) \exp(-i\omega c t - k\omega^2 t), \quad A(\omega): \text{unknown "constant"}$$

Initial condition: $u(x, 0) = f(x)$

$$U(\omega, 0) = \mathcal{L}[f(x)] = F(\omega)$$

$$U(\omega, 0) = A(\omega) \cdot \exp(0) = F(\omega) \Rightarrow A(\omega) = F(\omega)$$

$$U(\omega, t) = F(\omega) \underbrace{\exp(-k\omega^2 t)}_{H(\omega, t)} \exp(-i\omega c t)$$

Convolution theorem: $\mathcal{L}^{-1}[F(\omega)H(\omega)] = f(x) * h(x)$,

where $h(x) = \mathcal{L}^{-1}[H(\omega)]$

$H(\omega)$ is a gaussian $\rightarrow h(x) = \sqrt{\frac{\pi}{kt}} \exp(-x^2/4kt)$

So $\mathcal{L}^{-1}[F(\omega)H(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \sqrt{\frac{\pi}{kt}} \exp(-(x-s)^2/4kt) ds$

$$u(x, t) = \mathcal{L}^{-1}[U(\omega, t)] = \mathcal{L}^{-1}[F(\omega)H(\omega, t) \cdot \exp(-i\omega c t)]$$

Shift property $\Rightarrow \mathcal{L}^{-1}[U(\omega, t)] \cdot \mathbf{I}$ equals $\mathcal{L}^{-1}[F(\omega)H(\omega, t)]$
 evaluated at $x+ct$.

So solution is $\frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \sqrt{\frac{\pi}{kt}} \exp(-\frac{(x-s)^2}{4kt}) ds$ evaluated

(2)

at $x \rightarrow x+ct$

$$u(x,t) = \int_{-\infty}^{\infty} f(s) \frac{1}{\sqrt{4\pi kt}} \exp(-\frac{(x+ct-s)^2}{4kt}) ds.$$

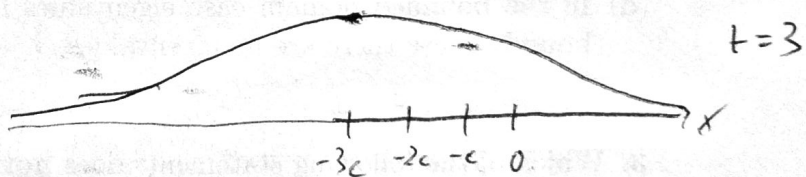
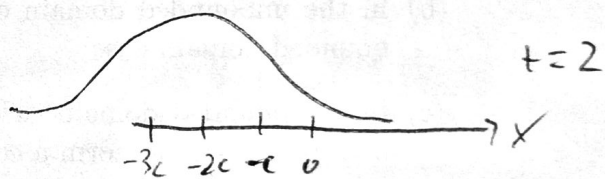
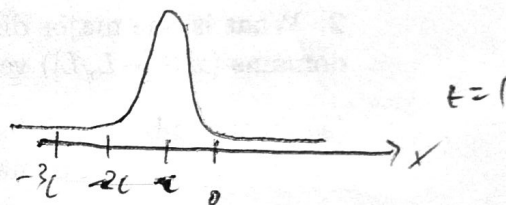
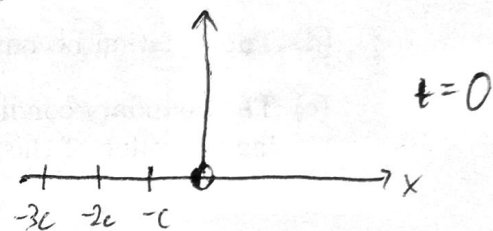
(b) $u(x,0) = f(x) = f(x)$

$$F(\omega) = \frac{1}{2\pi}$$

$$u(x,t) = \int_{-\infty}^{\infty} f(s) \frac{1}{\sqrt{4\pi kt}} \exp(-\frac{(x+ct-s)^2}{4kt}) ds$$

$$= \frac{1}{\sqrt{4\pi kt}} \exp(-\frac{(x+ct-0)^2}{4kt})$$

$$= \frac{1}{\sqrt{4\pi kt}} \exp(-\frac{(x+ct)^2}{4kt})$$



The convection term cu_x moves the solution to the left at speed c .

16.4.4

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$$(a) u_t = k u_{xx} - \gamma u, \quad -\infty < x < \infty$$

$$u(x, 0) = f(x)$$

$$U(\omega, t) = \mathcal{L}[u(x, t)]$$

$$U_t = k(-i\omega)^2 U - \gamma U \\ = (-\gamma - k\omega^2) U$$

$$U(\omega, t) = A(\omega) \exp(-k\omega^2 t - \gamma t), \quad A(\omega) \text{ an unknown "constant"}$$

$$\text{Initial data: } u(x, 0) = f(x) \Rightarrow U(x, 0) = \mathcal{L}[f(x)] = F(\omega)$$

$$U(\omega, 0) = A(\omega) \exp(0) = F(\omega) \Rightarrow A(\omega) = F(\omega)$$

$$U(\omega, t) = F(\omega) \exp(-k\omega^2 t - \gamma t)$$

$$= F(\omega) \exp(-k\omega^2 t) \underbrace{\exp(-\gamma t)}$$

is a constant with respect to $\omega \Rightarrow$ FT does not affect it.

$$u(x, t) = \mathcal{L}^{-1}[F(\omega) \exp(-k\omega^2 t) \exp(-\gamma t)]$$

$$= \exp(-\gamma t) \mathcal{L}^{-1}[F(\omega) \exp(-k\omega^2 t)]$$

$$= \exp(-\gamma t) \cdot (f * h)(x), \quad \text{where } h(x) = \mathcal{L}^{-1}[\exp(-k\omega^2 t)]$$

$$= \sqrt{\frac{\pi}{kt}} \exp(-x^2/4kt)$$

$$= \exp(-\gamma t) \cdot \underbrace{\int_{-\infty}^{\infty} f(s) \frac{1}{\sqrt{4\pi kt}} \exp(-|x-s|^2/4kt) ds}_{= f * h}$$

(b) The solution suggests that γ only plays a role in the $\exp(-\gamma t)$ factor, multiplying the solution to the standard heat equation.

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Then suppose we define $v(x,t) = \exp(\gamma t) u(x,t)$. What PDE does $v(x,t)$ satisfy?

$$\begin{aligned}\frac{\partial v}{\partial t} &= \frac{\partial}{\partial t} [\exp(\gamma t) \cdot u(x,t)] = \gamma \exp(\gamma t) \cdot u + \exp(\gamma t) \cdot u_t \\ &= \gamma \exp(\gamma t) \cdot u + \exp(\gamma t) [k u_{xx} - \gamma u] \\ &= k \exp(\gamma t) u_{xx} \\ &= k \frac{\partial^2}{\partial x^2} [\exp(\gamma t) u] \\ &= k v_{xx}\end{aligned}$$

$$\text{And } v(x,0) = \exp(\gamma \cdot 0) u(x,0) = f(x).$$

So $v(x,t)$ solves

$$v_t = k v_{xx}$$

$$v(x,0) = f(x),$$

which is a simpler equation. We could have first solved this and then computed u as $u(x,t) = \exp(-\gamma t) v(x,t)$.

16.4.6

$$\frac{d^2 y}{dx^2} - xy = 0, \quad y(x) = Ai(x)$$

$$\lim_{|x| \rightarrow \infty} y = 0, \quad y(0) = 3^{-2/3} / \Gamma(2/3) = \frac{1}{\pi} \int_0^{\infty} \cos\left(\frac{\omega^3}{3}\right) d\omega$$

Take Fourier transform of ODE: $(-i\omega)^2 Y(\omega) - (-i Y'(\omega)) = 0$

$$Y'(\omega) + i\omega^2 Y(\omega) = 0$$

$$Y''(\omega) + i\omega^2 Y(\omega) = 0$$

This is a first-order linear ODE \Rightarrow it can be solved via integrating factors.

$$Y'(\omega) + P(\omega)Y(\omega) = 0 \Rightarrow P(\omega) = i\omega^2$$

$$\begin{aligned} \text{integrating factor: } \mu(\omega) &= \exp\left(\int P(\omega) d\omega\right) \\ &= \exp\left(\frac{i}{3}\omega^3\right) \end{aligned}$$

(1) multiply ODE by $\mu(\omega)$

(2) manipulate left-hand side into a total derivative.

$$Y'(\omega) + i\omega^2 Y(\omega) = 0$$

$$Y'(\omega) \exp\left(\frac{i}{3}\omega^3\right) + Y(\omega) \cdot i\omega^2 \exp\left(\frac{i}{3}\omega^3\right) = 0$$

$$\frac{d}{d\omega} \left[Y(\omega) \exp\left(\frac{i}{3}\omega^3\right) \right] = 0$$

$$Y(\omega) \exp\left(\frac{i}{3}\omega^3\right) = c \quad (\text{undetermined constant})$$

$$Y(\omega) = c \cdot \exp\left(-\frac{i}{3}\omega^3\right)$$

How to compute c ?

$$y(0) = \frac{1}{\pi} \int_0^{\infty} \cos\left(\frac{\omega^3}{3}\right) d\omega$$

$$\text{By definition of FT: } y(0) = \int_{-\infty}^{\infty} Y(\omega) \exp(-i\omega \cdot 0) d\omega$$

$$= \int_{-\infty}^{\infty} Y(\omega) d\omega$$

$$= c \cdot \int_{-\infty}^{\infty} \exp\left(-\frac{i}{3}\omega^3\right) d\omega$$

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$$\begin{aligned} \frac{1}{\pi} \int_0^{\infty} \cos(\omega^3/3) d\omega = y(0) &= c \int_{-\infty}^{\infty} \exp(-\frac{i}{3}\omega^3) d\omega \\ &= c \cdot \left[\int_{-\infty}^{\infty} \cos(\omega^3/3) d\omega + i \int_{-\infty}^{\infty} \sin(\omega^3/3) d\omega \right] \\ &= c \left[2 \int_0^{\infty} \cos(\omega^3/3) d\omega + i \cdot 0 \right] \end{aligned}$$

↑
because $\cos(x)$
is an even
function
↑
because $\sin(x)$
is an odd function

$$\text{so } \frac{1}{\pi} \int_0^{\infty} \cos(\frac{\omega^3}{3}) d\omega = 2c \int_0^{\infty} \cos(\frac{\omega^3}{3}) d\omega$$

$$\implies c = \frac{1}{2\pi}$$

$$\mathcal{F}[A(x)] = Y(\omega) = \frac{1}{2\pi} \exp(-i\omega^3/3)$$

16.4.7 (a) $u_t = k u_{xxx}, \quad -\infty < x < \infty$

$$u(x,0) = f(x)$$

$$U(\omega, t) = \mathcal{F}[u(x, t)] \implies U_t = k(-i\omega)^3 U = ik\omega^3 U$$

$$U(\omega, t) = U(\omega, 0) \exp(ik\omega^3 t)$$

$$\text{Initial data} \implies U(\omega, 0) = \mathcal{F}[f(x)] = F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$$

$$U(\omega, t) = F(\omega) \exp(ik\omega^3 t)$$

$$\begin{aligned} u(x, t) &= \mathcal{F}^{-1}[U(\omega, t)] = \mathcal{F}^{-1}[F(\omega) \exp(ik\omega^3 t)] \\ &= \int_{-\infty}^{\infty} e^{ik\omega^3 t} e^{-i\omega x} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) e^{i\omega s} ds d\omega \end{aligned}$$

(b) Convolution theorem:

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$$u = F \cdot \exp(ikt + w^3)$$

$$\Rightarrow u(x, t) = (f * g)(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) g(x-s) ds$$

$$\text{where } g(x, t) = \mathcal{F}^{-1}[\exp(ikt + w^3)]$$

$$= \int_{-\infty}^{\infty} \exp(ikt + w^3 - iw x) dw$$

(c) Exercise 16.4.6 $\Rightarrow \mathcal{F}[Ai(x)] = \frac{1}{2\pi} \exp(-iw^3/3)$

$$\text{so } \mathcal{F}^{-1}[\exp(-\frac{i}{3}w^3)] = 2\pi Ai(x)$$

Define $Y(w) = \frac{1}{2\pi} \exp(-iw^3/3)$ (FT of Airy function)

Define $G(w) = \exp(ikt + w^3)$ (FT of g).

Goal: relate G to Y , in order to compute $g(x, t)$.

Assume $k > 0$: ~~Assume $k < 0$~~

$$2\pi Y(-(3kt)^{1/3}w) = \exp(-\frac{i}{3}(-(3kt)^{1/3}w)^3)$$

$$= \exp(-\frac{i}{3} \cdot (-3kt)w^3) = G(w)$$

$$\text{so } \mathcal{F}^{-1}[G(w)] = 2\pi \mathcal{F}^{-1}[Y(-(3kt)^{1/3}w)]$$

$$= 2\pi \int_{-\infty}^{\infty} Y(-(3kt)^{1/3}w) \cdot \exp(-iw x) dw$$

$$u = -(3kt)^{1/3}w$$

$$du = -(3kt)^{1/3} dw$$

$$= \frac{-2\pi}{(3kt)^{1/3}} \int_{\infty}^{-\infty} Y(u) \exp(-iu \left(\frac{-x}{(3kt)^{1/3}}\right)) du$$

$$= \frac{2\pi}{(3kt)^{1/3}} \int_{-\infty}^{\infty} Y(u) \exp(-iu \left(\frac{-x}{(3kt)^{1/3}}\right)) du$$

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$$\mathcal{L}^{-1}[G(\omega)] = \frac{2\pi}{(3kt)^{1/3}} \int_{-\infty}^{\infty} Y(u) \exp(-iu \left(\frac{-x}{(3kt)^{1/3}}\right)) du$$

$$= \frac{2\pi}{(3kt)^{1/3}} Y\left(\frac{-x}{(3kt)^{1/3}}\right)$$

$$= \frac{2\pi}{(3kt)^{1/3}} \text{Ai}\left[\frac{-x}{(3kt)^{1/3}}\right] = g(x,t).$$

So $u(x,t) = f * g$

$$= \int_{-\infty}^{\infty} f(s) \frac{1}{(3kt)^{1/3}} \text{Ai}\left[\frac{-(x-s)}{(3kt)^{1/3}}\right] ds$$