MATH 3150 HW #8 solutions

Section 16.3 # 1, 2, 3, 5, 6, 11

16.3.1 $\mathcal{F}[f(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx$

$F(\omega) = \mathcal{F}[f(x)], \quad G(\omega) = \mathcal{F}[g(x)]$

(a) $$\mathcal{F}[c_1 f(x) + c_2 g(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} (c_1 f(x) + c_2 g(x))\, e^{i\omega x}\, dx$$
$$= c_1 \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx + c_2 \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x) e^{i\omega x}\, dx$$
$$= c_1 \mathcal{F}[f(x)] + c_2 \mathcal{F}[g(x)]$$

(b) $$\mathcal{F}[f(x)g(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) g(x)\, e^{i\omega x}\, dx$$
$$F(\omega)\cdot G(\omega) = \frac{1}{4\pi^2}\left[\int_{\infty}^{\infty} f(x) e^{i\omega x}\, dx\right]\cdot\left[\int_{\infty}^{\infty} g(x) e^{i\omega x}\, dx\right]$$

These are not the same expression: $\mathcal{F}[f(x)g(x)] \neq F(\omega)\cdot G(\omega)$

16.3.2 $\mathcal{F}^{-1}[F(\omega)] = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x}\, d\omega$

(a) $$\mathcal{F}^{-1}[c_1 F(\omega) + c_2 G(\omega)] = \int_{-\infty}^{\infty} (c_1 F(\omega) + c_2 G(\omega))\, e^{-i\omega x}\, d\omega$$
$$= c_1 \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x}\, d\omega + c_2 \int_{-\infty}^{\infty} G(\omega) e^{-i\omega x}\, d\omega$$
$$= c_1 \mathcal{F}^{-1}[F(\omega)] + c_2 \mathcal{F}^{-1}[G(\omega)]$$

(b) $$\mathcal{F}^{-1}[F(\omega) G(\omega)] = \int_{-\infty}^{\infty} F(\omega) G(\omega)\, e^{-i\omega x}\, d\omega$$
$$f(x)\cdot g(x) = \left[\int_{-\infty}^{\infty} F(\omega) e^{-i\omega x}\, d\omega\right]\cdot\left[\int_{-\infty}^{\infty} G(\omega) e^{-i\omega x}\, d\omega\right]$$

These are not the same expression: $\mathcal{F}^{-1}[F(\omega) G(\omega)] \neq f(x) g(x)$

16.3.3 $F(\omega) = \mathcal{F}[f(x)]$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i\omega x} dx \qquad (e^{ix} = \cos x + i\sin x)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\cos x\,dx + \frac{i}{2\pi}\int_{-\infty}^{\infty} f(x)\sin x\,dx$$

so $F^*(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\cos x\,dx - \frac{i}{2\pi}\int_{-\infty}^{\infty} f(x)\sin x\,dx$ (since $f(x)$ is real, $f^*(x) = f(x)$)

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i(-\omega)x} dx = F(-\omega)$$

16.3.5 $F(\omega) = \mathcal{F}[f(x)]$

$f(x) = \mathcal{F}^{-1}[F(\omega)] = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega$

$\mathcal{F}^{-1}[e^{i\omega\beta}F(\omega)] = \int_{-\infty}^{\infty} F(\omega) e^{i\omega\beta} e^{-i\omega x} d\omega$

$$= \int_{-\infty}^{\infty} F(\omega) e^{-i\omega(x-\beta)} d\omega$$

$$= f(x-\beta)$$

16.3.6 $f(x) = \begin{cases} 0, & |x| > a \\ 1, & |x| < a \end{cases}$

Note: $e^{ix} = \cos x + i\sin x$

$\Longrightarrow \sin x = \frac{1}{2i}[e^{ix} - e^{-ix}]$

$$\mathcal{F}[f(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i\omega x} dx = \frac{1}{2\pi}\int_{-a}^{a} 1\cdot e^{i\omega x} dx$$

$$= \frac{1}{2\pi i\omega} e^{i\omega x}\Big|_{x=-a}^{a}$$

$$= \frac{1}{2\pi i\omega}\left[e^{i\omega a} - e^{-i\omega a}\right] = \frac{\sin(\omega a)}{\pi\omega}$$

16.3.11 (a) Assume $f(x)$ is a function with unit area, i.e.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Then consider, for $\alpha > 0$,

$$\int_{-\infty}^{\infty} \frac{1}{\alpha} f(x/\alpha)\,dx = \int_{-\infty}^{\infty} f(u)\,du = 1$$

$$u = x/\alpha$$
$$du = \frac{1}{\alpha}dx$$

So $\frac{1}{\alpha} f(x/\alpha)$ also has unit area.

(b) $F(\omega) = \mathcal{F}[f(x)]$

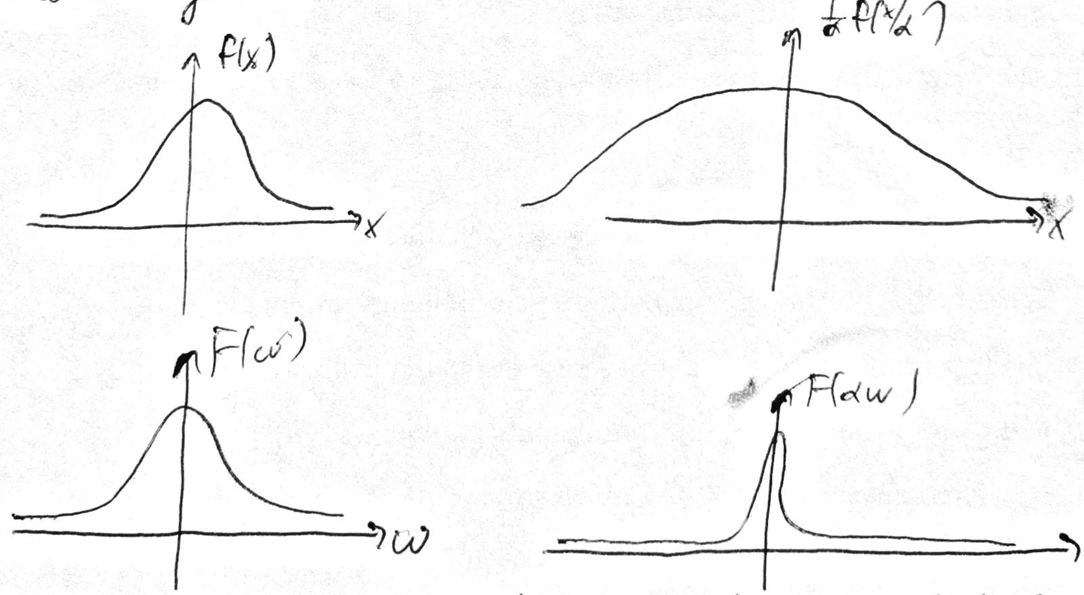
$$F(\alpha\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\alpha\omega x}\,dx = \frac{1}{2\pi}\cdot\frac{1}{\alpha}\int_{-\infty}^{\infty} f(u/\alpha)\, e^{i\omega u}\,du$$

$$u = \alpha x \qquad du = \alpha\,dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{\alpha} f(u/\alpha)\right] e^{i\omega u}\,du$$

$$= \mathcal{F}\left[\frac{1}{\alpha} f(x/\alpha)\right].$$

(c) Consider large α:



Thus spread-out functions ($\frac{1}{\alpha} f(x/\alpha)$) have sharply-peaked fourier transforms ($F(\alpha\omega)$). Taking small values of α shows the reverse relation: sharply-peaked functions have spread-out Fourier transforms

16.3.7 $F(\omega) = e^{-|\omega|\alpha}$, $\alpha > 0$

Compute $\mathcal{F}^{-1}\{F\}$

By definition: $\mathcal{F}^{-1}\{F(\omega)\} = \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega x}\, d\omega$

$$= \int_{-\infty}^{\infty} e^{-|\omega|\alpha} e^{-i\omega x}\, d\omega$$

$$= \int_{-\infty}^{0} e^{w\alpha} e^{-iwx}\, dw + \int_{0}^{\infty} e^{-w\alpha} e^{-iwx}\, dw$$

$$= \int_{-\infty}^{0} e^{w(\alpha - ix)}\, dw + \int_{0}^{\infty} e^{w(-\alpha - ix)}\, dw$$

$$= \frac{1}{\alpha - ix} e^{w(\alpha - ix)}\Big|_{-\infty}^{0} + \frac{1}{-\alpha - ix} e^{w(-\alpha - ix)}\Big|_{0}^{\infty}$$

$$= \frac{1}{\alpha - ix}\Big[e^{0} - \underbrace{\lim_{b\to\infty} e^{-b\alpha} e^{ibx}}_{0,\ \text{since } \alpha > 0}\Big] + \frac{1}{-\alpha - ix}\Big[\overbrace{\lim_{b\to\infty} e^{-b\alpha} e^{-ibx}}^{0,\ \text{since } \alpha > 0} - e^{0}\Big]$$

$$= \frac{1}{\alpha - ix} 1 - \frac{1}{\alpha + ix}(-1) = \frac{1}{\alpha - ix} + \frac{1}{\alpha + ix}$$

$$= \frac{2\alpha}{(\alpha - ix)(\alpha + ix)} = \frac{2\alpha}{\alpha^2 + x^2}$$