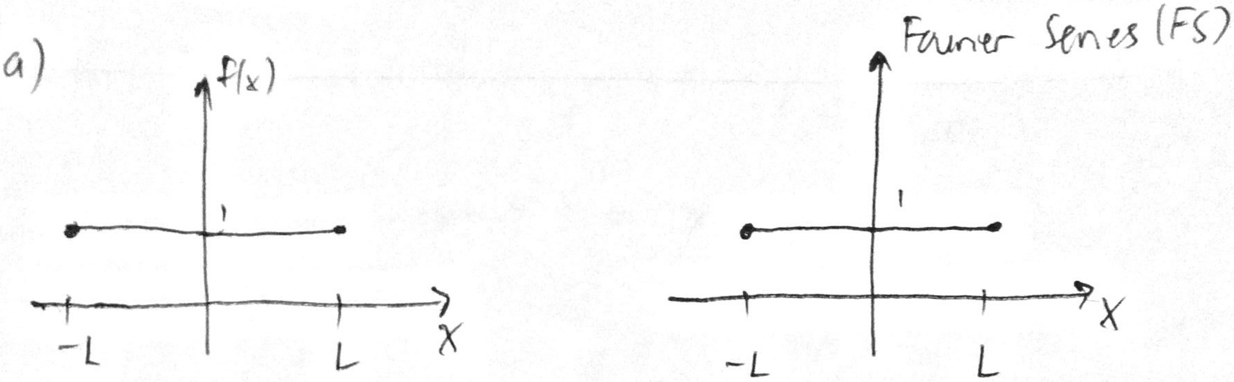
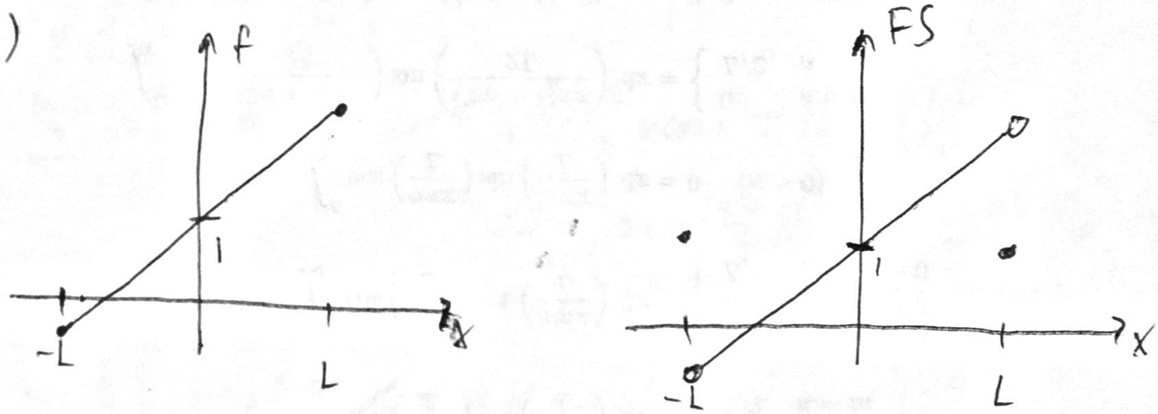
MATH 3150 HW #6 solutions

14.2.1 (a)



f(x) and its Fourier Series are equal.

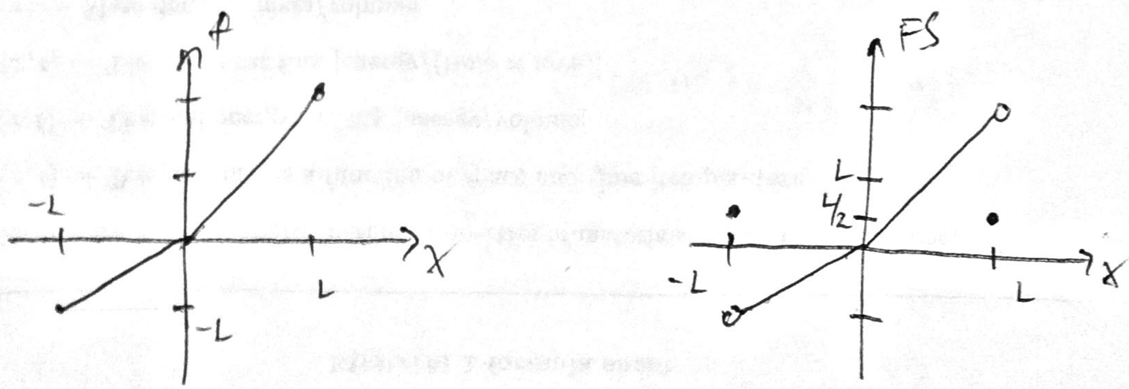
(c)



(As pictured, $L > 1$)

f and its Fourier Series are equal on the interior $(-L, L)$, but differ at $x = \pm L$.

(e)



f and its Fourier Series are equal on the interior $(-L, L)$, but differ at $x = \pm L$.

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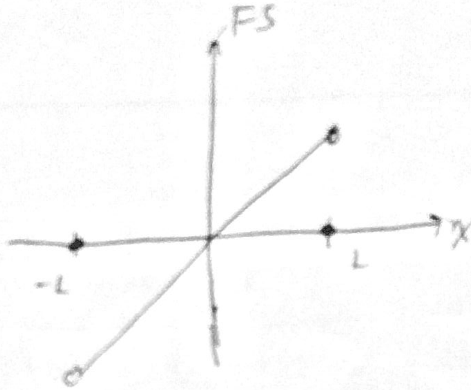
14.2.2 (a) $f(x) = x$



$$FS = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad (n>0)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} x\,dx = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} x\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left[\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx + \int_{-L}^{0} x\cos\left(\frac{n\pi x}{L}\right)dx\right]$$

$$= \frac{1}{L}\left[\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx - \int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx\right]$$

$$= 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} x\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx$$

$u = x \qquad v = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$

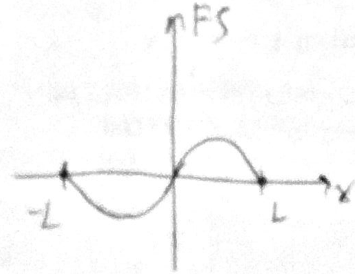
$du = dx \qquad dv = \sin\left(\frac{n\pi x}{L}\right)dx$

$$= \frac{2}{L}\left[-\frac{L}{n\pi}x\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{L}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx\right]$$

$$= \frac{2}{L}\left[-\frac{L^2}{n\pi}\cos(n\pi) + \frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L\right]$$

$$= \frac{2L}{n\pi}(-1)^{n+1}$$

(c) $f(x) = \sin\left(\frac{\pi x}{L}\right)$



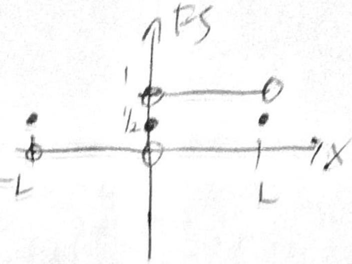
$$a_0 = \frac{1}{2L}\int_{-L}^{L}\sin\left(\frac{\pi x}{L}\right)dx = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L}\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = 0$$

(see midterm 1 formula sheet)

$$b_n = \frac{1}{L}\int_{-L}^{L}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases}0, & n \neq 1\\ 1, & n = 1\end{cases}$$

(f) $f(x) = \begin{cases} 0, & x<0 \\ 1, & x>0 \end{cases}$



$\underline{a_0} = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_0^L dx = \underline{\frac{1}{2}}$

$\underline{a_n} = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L = \underline{0}$

$\underline{b_n} = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx = -\frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L$

$= -\frac{1}{n\pi}\left[(-1)^n - 1\right]$

$= \underline{\frac{1}{n\pi}\left[1-(-1)^{n+1}\right]}$

14.2.3 FS of $f_1(x)$ is $\sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \cancel{dx} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

FS of $f_2(x)$ is $\sum_{n=0}^{\infty} e_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{L}\right)$

FS of $c_1 f_1(x) + c_2 f_2(x)$ is $\sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$

where: $a_n = \begin{cases} \frac{1}{2L}\int_{-L}^{L} f_1(x)\,dx, & n=0 \\ \frac{1}{L}\int_{-L}^{L} f_1(x)\cos\left(\frac{n\pi x}{L}\right)dx, & n>0 \end{cases}$ $\quad b_n = \frac{1}{L}\int_{-L}^{L} \cancel{\cos} f_1(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$e_n = \begin{cases} \frac{1}{2L}\int_{-L}^{L} f_2(x)\,dx, & n=0 \\ \frac{1}{L}\int_{-L}^{L} f_2(x)\cos\left(\frac{n\pi x}{L}\right)dx, & n>0 \end{cases}$ $\quad d_n = \frac{1}{L}\int_{-L}^{L} f_2(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$A_n = \begin{cases} \frac{1}{2L}\int_{-L}^{L} \left(c_1 f_1(x) + c_2 f_2(x)\right)dx, & n=0 \\ \frac{1}{L}\int_{-L}^{L} \left(c_1 f_1(x) + c_2 f_2(x)\right)\cos\left(\frac{n\pi x}{L}\right)dx, & n>0 \end{cases}$

$B_n = \frac{1}{L}\int_{-L}^{L} \left(c_1 f_1(x) + c_2 f_2(x)\right)\sin\left(\frac{n\pi x}{L}\right)dx$

By linearity of integral: $A_n = c_1 a_n + c_2 e_n$, $B_n = c_1 b_n + c_2 d_n$

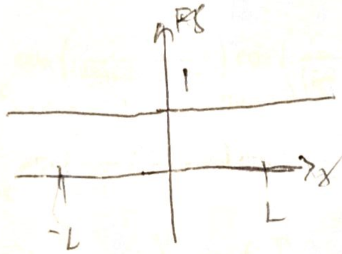
Thus,

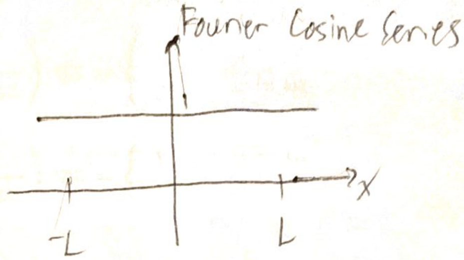
$$c_1(\text{FS of } f_1) + c_2(\text{FS of } f_2)$$

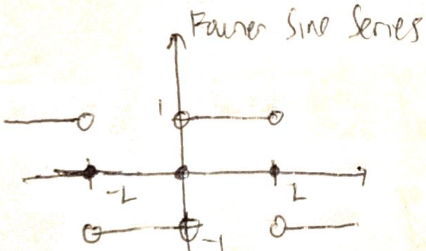
$$= c_1\left[\sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\right] + c_2\left[\sum_{n=0}^{\infty} e_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$= \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = (\text{FS of } c_1 f_1(x) + c_2 f_2(x)).$$

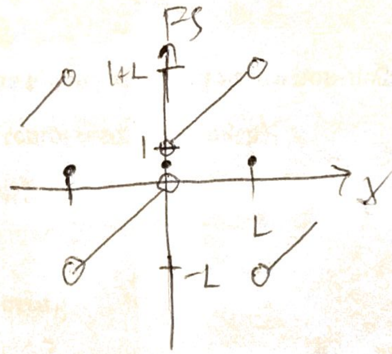
14.3.1 (a) $f(x) = 1$

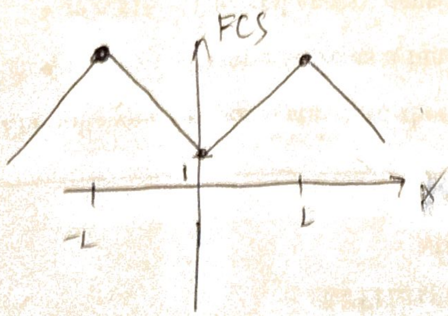


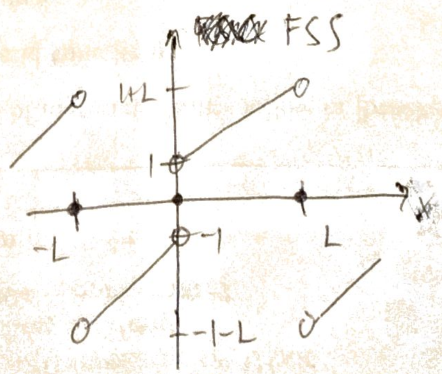




(c) $f(x) = \begin{cases} x, & x<0 \\ 1+x, & x>0 \end{cases}$







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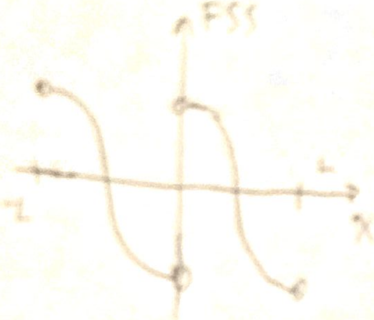
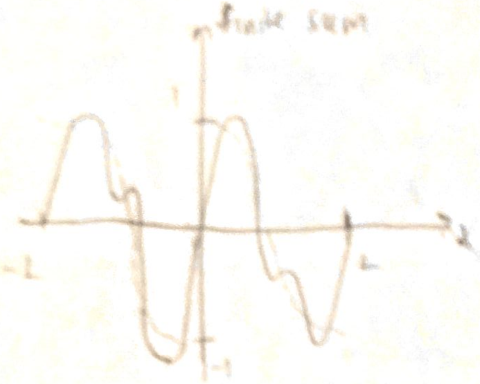
4.3.3 (a) $f(x) = \cos(\pi x/L)$





(c) $f(x) = x$