MATH 3150 HW #4 Solutions

13.3.7 $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial u}{\partial x}(0,t)=0$ $\frac{\partial u}{\partial x}(L,t)=0$

$u(x,0)=f(x)$

(a) This problem models temperature distribution in a one-dimensional rod whose endpoints are insulated with initial temperature distribution $f(x)$.

(b) Separation of variables: $u(x,t)=\phi(x)T(t)$

$$u_t = k u_{xx} \longrightarrow T'(t)\phi(x) = k\,T(t)\phi''(x)$$

$$\frac{T'(t)}{kT(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda \quad \text{unknown constant}$$

$\frac{\partial u}{\partial x}(0,t)=0 \Rightarrow \phi'(0)=0$

$\frac{\partial u}{\partial x}(L,t)=0 \Rightarrow \phi'(L)=0$

ODE for ϕ: $\begin{cases} \phi''(x)+\lambda\phi(x)=0 \\ \phi'(0)=0, \quad \phi'(L)=0 \end{cases}$

$\underline{\lambda<0}$ $\phi''(x) - |\lambda|\phi(x) = 0$

$r^2 - |\lambda| = 0 \Longrightarrow r = \pm\sqrt{|\lambda|}$

$r_1 = \sqrt{|\lambda|}$, $r_2 = -\sqrt{|\lambda|}$ and $r_1 \neq r_2$

$\phi(x) = c_1\exp(r_1x) + c_2\exp(r_2x)$

$\phi'(0)=0 \Rightarrow c_1r_1 + c_2r_2 = 0$

$c_1r_1 - c_2r_1 = 0$

$r_1 \neq 0 \Rightarrow c_1 = c_2$

$\phi'(L)=0 \Rightarrow c_1r_1\exp(r_1L) + c_2r_2\exp(r_2L)=0$

$c_1r_1[\exp(r_1L) - \exp(r_2L)] = 0$

$r_1\neq 0$, and $\exp(r_1L)\neq\exp(r_2L)$, so $c_1=0 \Rightarrow c_2=0$

Thus only $\phi(x)=0$ is a solution.

$\underline{\lambda=0}$ $\quad \phi''(x)=0$

$\phi(x)=c_1+c_2x$, $\phi'(x)=c_2$

$\phi'(0)=0 \Rightarrow c_2=0$

$\phi'(L)=0 \Rightarrow c_2=0$

$\phi(x)=c_1$ is a solution, nontrivial if $c_1 \neq 0$.

Let's call this nontrivial solution $\phi_0(x)=c_1$, $\lambda=0$

If $\lambda=0 \rightarrow T_0'(t)=-\lambda k T_0(t)=0$

$T_0(t)=c_2$

So this solution is $u_0(x,t)=\phi_0(x)\,T_0(t)=c$, for $c\neq 0$.

$\underline{\lambda>0}$ $\quad \phi''+|\lambda|\phi=0$

$r^2+|\lambda|=0 \Rightarrow r_1=i\sqrt{|\lambda|}$, $r_2=-i\sqrt{|\lambda|}$

$\phi(x)=c_1\cos(x\sqrt{|\lambda|})+c_2\sin(x\sqrt{|\lambda|})$

$\phi'(x)=c_1\sqrt{|\lambda|}\,(-\sin(x\sqrt{|\lambda|}))+c_2\sqrt{|\lambda|}\cos(x\sqrt{|\lambda|})$

$\phi'(0)=0 \Rightarrow c_2\sqrt{|\lambda|}=0 \Rightarrow c_2=0$

$\phi'(L)=0 \Rightarrow -c_1\sqrt{|\lambda|}\sin(L\sqrt{|\lambda|})=0 \Longrightarrow \sin(L\sqrt{|\lambda|})=0$

$L\sqrt{|\lambda|}=n\pi$, $n=1,2,3,\ldots$

$\lambda_n=\left(\frac{n\pi}{L}\right)^2$

$\lambda=\lambda_n=\left(\frac{n\pi}{L}\right)^2 \Rightarrow \phi_n(x)=\cos\left(\frac{n\pi x}{L}\right)$

$\lambda=\lambda_n \Rightarrow T_n'(t)=-\lambda_n k T_n(t)$

$T_n(t)=c_n\cdot\exp(-\lambda_n k t)$

These solutions are $u_n(x,t)=\phi_n(x)\,T_n(t)=c_n\exp(-\lambda_n k t)\cos\left(\frac{n\pi x}{L}\right)$

$\lambda_n=\left(\frac{n\pi}{L}\right)^2$

For $\lambda=0$, $u_0(x,t)$ does not grow in time

For $\lambda>0$, $u_n(x,t)$ decays exponentially in time

So no separated solutions grow exponentially in time.

(c) General solution is $u(x,t)=u_0(x,t)+\sum_{n=1}^{\infty} u_n(x,t)$

$$= A_0 + \sum_{n=1}^{\infty} A_n \exp\left(-k\left(\tfrac{n\pi}{L}\right)^2 t\right)\cos\left(\tfrac{n\pi x}{L}\right)$$

This satisfies initial condition of $u(x,0)=f(x)$

$$A_0+\sum_{n=1}^{\infty} A_n \cos\left(\tfrac{n\pi x}{L}\right)=f(x)$$

(d) Use $\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0, & n\neq m \\ L/2, & n=m\neq 0 \\ L, & n=m=0. \end{cases}$

$$A_0+\sum_{n=1}^{\infty} A_n \cos\left(\tfrac{n\pi x}{L}\right)=f(x)$$

$\downarrow$ multiply by $\cos\left(\frac{m\pi x}{L}\right)$, integrate

$$A_0 L = \int_0^L f(x)\,dx \quad (\text{if } m=0)$$

$$A_m \tfrac{L}{2} = \int_0^L f(x)\cos\left(\tfrac{m\pi x}{L}\right)dx \quad (\text{if } m\neq 0)$$

$$A_0 = \tfrac{1}{L}\int_0^L f(x)\,dx$$

$$A_m = \tfrac{2}{L}\int_0^L f(x)\cos\left(\tfrac{m\pi x}{L}\right)dx$$

(e) the equilibrium/steady-state solution of the PDE is $u(x)=\frac{1}{L}\int_0^L f(x)dx$.

Our solution is $u(x,t)=A_0+\sum_{n=1}^{\infty} A_n \exp\left(-k\left(\frac{n\pi}{L}\right)^2 t\right)\cos\left(\frac{n\pi x}{L}\right)$

Taking $t\to\infty$: $\lim_{t\to\infty} u(x,t) = A_0 = \frac{1}{L}\int_0^L f(x)dx$ ✓.

13.3.8 $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \alpha u$

$u(0,t)=0, \quad u(L,t)=0.$

(a) $\alpha>0$, $\frac{\partial u}{\partial t}=0 \implies u''(x) - \frac{\alpha}{k}u(x) = 0$, $\frac{\alpha}{k}>0.$

$$r^2 - \frac{\alpha}{k} = 0$$

$$r_1 = \sqrt{\frac{\alpha}{k}}, \quad r_2 = -\sqrt{\frac{\alpha}{k}}, \quad r_1\neq r_2$$

$$u(x) = c_1\exp(r_1x) + c_2\exp(r_2x)$$

$$u(0)=0 \implies c_1+c_2=0, \text{ so } c_1=-c_2$$

$$u(L)=0 \implies c_1\exp(r_1L) - c_1\exp(r_2L) = 0$$

$$r_1\neq r_2 \implies \exp(r_1L)-\exp(r_2L)\neq 0 \implies c_1=0,\ c_2=0$$

Equilibrium solution is $u(x)=0$.

(b) Separation of variables: $u(x,t) = \phi(x)T(t)$

$$u_t = ku_{xx} - \alpha u$$

$$\frac{T'(t)}{kT(t)} = \frac{\phi''(x)}{\phi(x)} - \frac{\alpha}{k} = -\lambda \text{ (unknown constant)}$$

ODE for ϕ: $\begin{cases} \phi''(x) + (\lambda - \frac{\alpha}{k})\phi(x) = 0 \\ \phi(0)=0, \quad \phi(L)=0 \end{cases}$

We already know the nontrivial solutions for this problem:

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \text{ with } \left(\lambda_n - \frac{\alpha}{k}\right) = \left(\frac{n\pi}{L}\right)^2$$

$$\lambda_n = \frac{\alpha}{k} + \left(\frac{n\pi}{L}\right)^2$$

For $\lambda = \lambda_n$: $T_n'(t) = -k\lambda_n T_n(t)$

$$= -\left(\alpha + K\left(\frac{n\pi}{L}\right)^2\right) T_n(t)$$

$$T_n(t) = \exp\left(-\left(\alpha + k\left(\frac{n\pi}{L}\right)^2\right)t\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \phi_n(x) T_n(t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\left(\alpha + k\left(\frac{n\pi}{L}\right)^2\right)t\right)$$

Since $\alpha + k\left(\frac{n\pi}{L}\right)^2 = 0$, then $\lim_{t\to\infty} u(x,t) = 0$, which matches (a).

<u>13.4.1</u> $u_t = ku_{xx}$, $0 < x < L$, $t > 0$

$\frac{\partial u}{\partial t}(0,t) = 0$, $\frac{\partial u}{\partial t}(L,t) = 0$

$u(x,0) = f(x)$.

Separation of variables: $u(x,t) = \phi(x) T(t)$

$$u_t = ku_{xx}$$

$$\frac{T'(t)}{kT(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda$$

$$\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi'(0) = 0, \quad \phi'(L) = 0 \end{cases}$$

Table in book $\Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2$, $\phi_n(x) = \cos\left(\frac{n\pi x}{L}\right)$, $n = 0, 1, 2 \ldots$

$T_n'(t) = -\lambda_n k T_n(t)$

$$T_n(t) = \exp(-k\lambda_n t) = \exp\left(-k\left(\frac{n\pi}{L}\right)^2 t\right)$$

$$u(x,t) = \sum_{n=0}^{\infty} A_n \exp\left(-k\left(\frac{n\pi}{L}\right)^2 t\right) \cos\left(\frac{n\pi x}{L}\right)$$

@ $t=0$: $f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$

Orthogonality: $\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0, & m \neq n \\ L, & m = n = 0 \\ L/2, & m = n \neq 0 \end{cases}$

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \quad + \quad \text{orthogonality} \Longrightarrow A_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx\,, \quad n>0.$$

(a) $u(x,0) = \begin{cases} 0, & x < L/2 \\ 1, & x > L/2 \end{cases}$

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_{L/2}^L 1\,dx = \frac{1}{2}$$

$A_n =$

$$= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{L/2}^{L} \cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\cdot\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{L/2}^{L}$$

$$= \frac{2}{n\pi}\left[\sin(n\pi) - \sin\left(n\frac{\pi}{2}\right)\right]$$

$$= \frac{2}{n\pi}\left(0 - \sin\left(\frac{n\pi}{2}\right)\right)$$

$$= \begin{cases} 0, & n \text{ even} \\ (-1)^{\left(\frac{n+1}{2}\right)}\dfrac{2}{n\pi}, & n \text{ odd} \end{cases}$$

$$u(x,t) = \sum_{n=0}^{\infty} A_n \exp(-k\lambda_n t)\cos\left(\frac{n\pi x}{L}\right)$$

$$= \frac{1}{2} + \sum_{n \text{ odd}} (-1)^{\left(\frac{n+1}{2}\right)} \frac{2}{n\pi}\exp(-k\lambda_n t)\cos\left(\frac{n\pi x}{L}\right)$$

$$\underline{u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2}{\pi(2n-1)}\exp\left(-k\left(\frac{(2n-1)\pi}{L}\right)^2 t\right)\cos\left(\frac{(2n-1)\pi x}{L}\right)}$$

(b) $u(x,0) = f(x) = 6 + 4\cos\left(\frac{3\pi x}{L}\right)$

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^L \left(6 + 4\cos\left(\frac{3\pi x}{L}\right)\right)dx$$

$$= 6$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L 6\cos\left(\frac{n\pi x}{L}\right)dx + \frac{2}{L}\int_0^L 4\cos\left(\frac{3\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \begin{cases} 0, & n \neq 3 \text{ (orthogonality)} \\ 4, & n = 3 \end{cases}$$

$$u(x,t) = \sum_{n=0}^{\infty} A_n \exp(-\lambda_n k t)\cos\left(\frac{n\pi x}{L}\right)$$

$$\underline{u(x,t) = 6 + 4\exp\left(-k\left(\frac{3\pi}{L}\right)^2 t\right)\cos\left(\frac{3\pi x}{L}\right)}$$

<u>13.4.3</u> $\phi'' + \lambda\phi = 0$

$\phi(0) = \phi(2\pi)$, $\phi'(0) = \phi'(2\pi)$

<u>$\lambda < 0$</u> $\phi(x) = c_1\exp(r_1 x) + c_2\exp(r_2 x)$, $r_1 = \sqrt{|\lambda|}$, $r_2 = -\sqrt{|\lambda|} = -r_1$

$\phi(0) = \phi(2\pi) \Rightarrow c_1 + c_2 = c_1\exp(r_1 2\pi) + c_2\exp(-r_1 2\pi)$

$c_1[1 - \exp(2\pi r_1)] + c_2[1 - \exp(-2\pi r_1)] = 0$

$\phi'(0) = \phi'(2\pi) \Rightarrow c_1 r_1 - c_2 r_1 = c_1 r_1\exp(2\pi r_1) - c_2 r_1\exp(-2\pi r_1)$

$c_1 r_1[1 - \exp(2\pi r_1)] + c_2 r_1[-1 + \exp(-2\pi r_1)] = 0$

Let $A = 1 - \exp(2\pi r_1)$
$B = 1 - \exp(-2\pi r_1)$ (note $A \neq B$)

$A \neq 0$ and $B \neq 0$ since $r_1 \neq 0$

Then our equations are $\begin{matrix} Ac_1 + Bc_2 = 0 \\ Ac_1 - Bc_2 = 0 \end{matrix} \rightarrow \underbrace{\begin{pmatrix} A & B \\ A & -B \end{pmatrix}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

determinant of this matrix is $-2AB \neq 0$

So the only solution is $c_1 = c_2 = 0$. $\rightarrow$ no negative λ are eigenvalues.

$\underline{\lambda = 0}$: $\phi(x) = c_1 + c_2 x$

$\phi(0) = \phi(2\pi) \Rightarrow c_1 = c_1 + 2\pi c_2 \Rightarrow c_2 = 0.$

$\phi'(0) = \phi'(2\pi) \Rightarrow c_2 = c_2 = 0$

So $\lambda_0 = 0$ with $\phi_0(x) = \underline{1}$ is an eigenvalue/eigenfunction

$\underline{\lambda > 0}$: $\phi(x) = c_1 \cos(x\sqrt{|\lambda|}) + c_2 \sin(x\sqrt{|\lambda|})$

$\phi(0) = \phi(2\pi) \Rightarrow c_1 = c_1 \cos(2\pi\sqrt{|\lambda|}) + c_2 \sin(2\pi\sqrt{|\lambda|})$

$\phi'(0) = \phi'(2\pi) \Rightarrow \sqrt{|\lambda|}\, c_2 = -c_1\sqrt{|\lambda|} \sin(2\pi\sqrt{|\lambda|}) + c_2\sqrt{|\lambda|} \cos(2\pi\sqrt{|\lambda|})$

Let $A = \cos(2\pi\sqrt{|\lambda|})$, $B = \sin(2\pi\sqrt{|\lambda|})$

Our equations are $\begin{matrix} c_1 = Ac_1 + Bc_2 \\ c_2 = -Bc_1 + Ac_2 \end{matrix} \rightarrow \begin{pmatrix} A-1 & B \\ -B & A-1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

determinant of matrix is $(A-1)^2 + B^2 = A^2 + B^2 + 1 - 2A$

$= 2 - 2A$

This equals 0 when $A = 1$

$\cos(2\pi\sqrt{|\lambda|}) = 1$

$\lambda_n = n^2, \quad n = 1, 2, \ldots$

If $\lambda = \lambda_n$: $A = \cos(2\pi n) = 1$

$B = \sin(2\pi n) = 0$

Then $\begin{pmatrix} A-1 & B \\ -B & A-1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

So c_1 and c_2 can be anything.

Thus $\lambda = \lambda_n \Rightarrow C_n =$ ~~...~~ $\cos(nx)$ is an eigenfunction

$S_n = \sin(nx)$ is an(other) eigenfunction

λ	ϕ
0	1
n^2 $(n>0)$	$\cos(nx)$
n^2 $(n>0)$	$\sin(nx)$.

13.4.4 $\phi'' + \lambda\phi = 0$

$\phi'(0) = 0, \quad \phi'(L) = 0.$

$\lambda < 0$: $\phi(x) = c_1\exp(r_1x) + c_2\exp(r_2x), \quad r_1 = \sqrt{|\lambda|}, \quad r_2 = -r_1 = -\sqrt{|\lambda|}$

$\phi'(x) = c_1r_1\exp(r_1x) - c_2r_1\exp(-r_1x)$

$\phi'(0) = 0 \Rightarrow c_1 - c_2 = 0 \Rightarrow c_1 = c_2$

$\phi'(L) = 0 \Rightarrow c_1\exp(r_1L) - c_2\exp(-r_1L) = 0$

$c_1 \underbrace{[\exp(r_1L) - \exp(-r_1L)]}_{\neq 0 \text{ since } r_1 \neq 0.} = 0$

So $c_1 = 0 \Rightarrow c_2 = 0 \Rightarrow \lambda < 0$ not an eigenvalue.