MATH 3150 Homework #3 solutions

13.3.1 (a) $\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$

suppose $u = u(t,r) = T(t)R(r)$

$$\text{PDE} \longrightarrow \frac{\partial}{\partial t}(T(t)R(r)) = \frac{k}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(T(t)R(r))\right]$$

$$R\cdot\frac{dT}{dt} = T\cdot\frac{k}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)$$

$$= T\cdot\frac{k}{r}\left(\frac{dR}{dr} + r\frac{d^2R}{dr^2}\right)$$

$$R\frac{dT}{dt} = k\cdot\frac{T}{r}\frac{dR}{dr} + T\cdot k\frac{d^2R}{dr^2}$$

$$\frac{T'(t)}{T} = k\cdot\frac{R'(r)}{r\cdot R} + k\cdot\frac{R''(r)}{R} = -\lambda$$

$\rightarrow$ constant because LHS is only a function of t, RHS is only a function of r. (negative sign is arbitrary)

$$\begin{cases} T'(t) + \lambda T(t) = 0 \\ R''(r) + \frac{1}{r}R'(r) + \frac{\lambda}{k}R(r) = 0 \end{cases}$$

(c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$u(x,y) = \phi(x)\psi(y)$$

$$\text{PDE} \longrightarrow \frac{\partial^2}{\partial x^2}(\phi(x)\psi(y)) + \frac{\partial^2}{\partial y^2}(\phi(x)\psi(y)) = 0$$

$$\phi''\cdot\psi + \phi\psi'' = 0$$

$$\frac{\phi''}{\phi} = -\frac{\psi''}{\psi} = \lambda \quad \text{(constant)}$$

$$\begin{cases} \phi''(x) - \lambda\phi(x) = 0 \\ \psi''(y) + \lambda\psi(y) = 0 \end{cases}$$

(e) $\frac{\partial u}{\partial t} = k \frac{\partial^4 u}{\partial x^2}$

$$u(x,t) = \varphi(x) T(t)$$

PDE $\rightarrow \frac{\partial}{\partial t}(\varphi T) = k \frac{\partial^4}{\partial x^4}(\varphi T)$

$$\varphi T' = k \varphi^{(4)} T$$

$$\frac{T'}{T} = k \frac{\varphi^{(4)}}{\varphi} = \lambda \text{ (constant)}$$

$$\begin{cases} T'(t) - \lambda T(t) = 0 \\ \varphi^{(4)}(x) - \frac{\lambda}{k}\varphi(x) = 0 \end{cases}$$

13.3.2 (a) $\varphi''(x) + \lambda \varphi(x) = 0$

$\varphi(0) = 0, \quad \varphi(\pi) = 0$

$\lambda < 0$: assume $\varphi(x) = \exp(rx)$ for r an unknown constant.

$$\varphi''(x) = r^2 \exp(rx)$$

ODE $\rightarrow r^2 \exp(rx) + \lambda \exp(rx) = 0$

"characteristic equation" $\rightarrow r^2 = -\lambda$ $\quad$ ($\exp(rx)$ never vanishes)

$r^2 = |\lambda|$ (since $\lambda < 0$)

$r = \pm\sqrt{|\lambda|}$. Let $r_1 = +\sqrt{|\lambda|}$, $r_2 = -\sqrt{|\lambda|}$

$$\varphi(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

$\varphi(0) = 0 \Rightarrow c_1 + c_2 = 0 \longrightarrow c_1 = -c_2$

$\varphi(\pi) = 0 \Rightarrow c_1 e^{\pi r_1} + c_2 e^{\pi r_2} = 0.$

$$c_2 \underbrace{\left[e^{\pi r_2} - e^{\pi r_1}\right]}_{\text{never vanishes since } r_1 = \sqrt{|\lambda|} \neq -\sqrt{|\lambda|} = r_2} = 0$$

$c_2 = 0 \Rightarrow c_1 = 0$

So there are no nontrivial solutions $\Rightarrow \lambda < 0$ not an eigenvalue

$\lambda = 0$: Characteristic equation: $r^2 = 0$

$r = 0$

$\varphi(x) = c_1 + c_2 x$

$\varphi(0) = 0 \Rightarrow c_1 = 0$

$\varphi(\pi) = 0 \Rightarrow c_2 \pi = 0 \rightarrow c_2 = 0$

Again no nontrivial solutions $\rightarrow \lambda = 0$ not an eigenvalue.

$\lambda > 0$ Characteristic equation: $r^2 = -\lambda = -|\lambda|$

$r = \pm i\sqrt{|\lambda|}$

$\varphi(x) = c_1 \cos(x\sqrt{|\lambda|}) + c_2 \sin(x\sqrt{|\lambda|})$

$\varphi(0) = 0 \Rightarrow c_1 = 0$

$\varphi(\pi) = 0 \Rightarrow c_2 \sin(\pi\sqrt{|\lambda|}) = 0$

c_2 can be nonzero if $\sin(\pi\sqrt{|\lambda|}) = 0$

$\sqrt{|\lambda|} = 1, 2, 3, \ldots$

Let $\lambda_n = n^2$ for $n = 1, 2 \ldots$

$\lambda = \lambda_n \Rightarrow \varphi(x) = \sin(x\sqrt{\lambda_n}) = \sin(nx)$ is a solution.

eigenvalues $\lambda_n = n^2$ $(n = 1, 2 \ldots)$, eigenfunctions $\varphi_n(x) = \sin(nx)$

(1b) $\varphi(0) = 0$, $\varphi(1) = 0$

$\varphi''(x) + \lambda\varphi(x) = 0$

$\lambda < 0$: characteristic equation $r^2 = -\lambda = |\lambda|$

$r = \pm\sqrt{|\lambda|}$,

$\varphi(x) = c_1 \exp(x\sqrt{|\lambda|}) + c_2 \exp(-x\sqrt{|\lambda|})$

$\varphi(0) = 0 \Rightarrow c_1 + c_2 = 0 \longrightarrow c_1 = -c_2$

$\varphi(1) = 0 \Rightarrow c_1 e^{\sqrt{|\lambda|}} + c_2 e^{-\sqrt{|\lambda|}} = 0$

$c_2(e^{-\sqrt{|\lambda|}} - e^{\sqrt{|\lambda|}}) = 0 \rightarrow c_2 = 0 \rightarrow c_1 = 0$

So no negative eigenvalues

$\underline{\lambda=0}$ $\quad \phi''(x)=0$

$\phi(x)=c_1+c_2x$

$\phi(0)=0 \implies c_1=0$

$\phi(1)=0 \implies c_2=0$

$\lambda=0$ not an eigenvalue

$\underline{\lambda>0}$ characteristic equation: $r^2=-\lambda=-|\lambda|$

$r=\pm i\sqrt{|\lambda|}$

$\phi(x)=c_1\cos(x\sqrt{|\lambda|})+c_2\sin(x\sqrt{|\lambda|})$

$\phi(0)=0 \Rightarrow c_1=0$

$\phi(1)=0 \Rightarrow c_2\sin(\sqrt{|\lambda|})=0$

$c_2\neq 0$ if $\sin(\sqrt{|\lambda|})=0 \rightarrow \lambda_n=(n\pi)^2, \ n=1,2,\ldots$

eigenvalues $\lambda_n=(n\pi)^2, n=1,2,\ldots$

eigenfunctions $\phi_n(x)=\sin(n\pi x)$

(d) $\phi''(x)+\lambda\phi(x)=0$

$\phi(0)=0, \quad \phi'(L)=0$

$\underline{\lambda<0}$: characteristic equation $r^2=-\lambda=|\lambda|$

$r=\pm\sqrt{|\lambda|}, \ r_1=+\sqrt{|\lambda|}, \ r_2=-\sqrt{|\lambda|}$

$\phi(x)=c_1e^{r_1x}+c_2e^{r_2x}$

$\phi(0)=0 \Rightarrow c_1+c_2=0 \longrightarrow c_1=-c_2$

$\phi'(L)=0 \Rightarrow c_1r_1e^{r_1L}+c_2r_2e^{r_2L}=0$

$c_2(r_2e^{r_2L}-r_1e^{r_1L})=0$

$c_2(r_2e^{r_2L}+r_2e^{r_1L})=0$ (since $r_1=-r_2$)

$$c_2 r_2 \underbrace{(e^{r_2 L} + e^{r_1 L})} = 0$$

$r_2 \neq 0$ sum of positive numbers $\neq 0$

So $c_2 = 0 \rightarrow c_1 = 0$

$\lambda < 0$ not an eigenvalue

$\underline{\lambda = 0}$ $\varphi''(x) = 0$

$\varphi(x) = c_1 + c_2 x$

$\varphi(0) = 0 \Rightarrow c_1 = 0$

$\varphi'(L) = 0 \Rightarrow c_2 = 0$

$\lambda = 0$ not an eigenvalue

$\underline{\lambda > 0}$ characteristic equation: $r^2 = -\lambda = -|\lambda|$

$r = \pm i\sqrt{|\lambda|}$

$\varphi(x) = c_1 \cos(x\sqrt{|\lambda|}) + c_2 \sin(x\sqrt{|\lambda|})$

$\varphi(0) = 0 \Rightarrow c_1 = 0$

$\varphi'(L) = 0 \Rightarrow c_2\sqrt{|\lambda|}\cos(L\sqrt{|\lambda|}) = 0$

c_2 can be nonzero if $\cos(L\sqrt{|\lambda|}) = 0$

$L\sqrt{|\lambda|} = (2n-1)\frac{\pi}{2}$, $n = 1, 2 \ldots$

$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2$, $n = 1, 2 \ldots$

eigenvalues $\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2$, $n = 1, 2 \ldots$

eigenfunctions $\varphi_n(x) = \sin\left(x \cdot \frac{(2n-1)\pi}{2L}\right)$.

<u>13.3.3</u> $u_t = k u_{xx}$ $(k > 0)$

$u(0,t) = 0$, $u(L,t) = 0$

Before using initial data, find eigenfunctions via separation of variables.

assume $u(x,t) = \varphi(x)\,T(t)$

$$u_t = k u_{xx} \longrightarrow \frac{T'(t)}{T(t)} = \frac{k\varphi''(x)}{\varphi(x)} = -\lambda \text{ , (constant)}$$

$$\varphi''(x) + \frac{\lambda}{k}\varphi(x) = 0$$

$u(0,t)=0 \Rightarrow \varphi(0)=0$

$u(L,t)=0 \Rightarrow \varphi(L)=0$

$\lambda/k < 0 \Rightarrow \varphi(x) = c_1 \exp\left(x\sqrt{\left|\frac{\lambda}{k}\right|}\right) + c_2 \exp\left(-x\sqrt{|\lambda/k|}\right)$

$\varphi(0)=0 \Rightarrow c_1 + c_2 = 0 \longrightarrow c_1 = -c_2$

$\varphi(L)=0 \Rightarrow c_1 \exp\left(L\sqrt{|\lambda/k|}\right) + c_2 \exp\left(-L\sqrt{|\lambda/k|}\right) = 0$

$c_2 \underbrace{\left[\exp\left(-L\sqrt{|\lambda/k|}\right) - \exp\left(L\sqrt{|\lambda/k|}\right)\right]}_{\neq 0 \text{ since exponents differ}} = 0$

$c_2 = 0 \rightarrow c_1 = 0 \rightarrow \lambda < 0$ not an eigenvalue

$\lambda/k = 0 \Rightarrow \varphi(x) = c_1 + c_2 x$

$\varphi(0)=0 \Rightarrow c_1 = 0$

$\varphi(L)=0 \Rightarrow c_2 = 0$

$\lambda/k = 0$ not an eigenvalue.

$\lambda/k > 0 \Rightarrow \varphi(x) = c_1 \cos\left(x\sqrt{|\lambda|/k}\right) + c_2 \sin\left(x\sqrt{|\lambda|/k}\right)$

$\varphi(0)=0 \Rightarrow c_1 = 0$

$\varphi(L)=0 \Rightarrow c_2 \sin\left(L\sqrt{|\lambda|/k}\right) = 0$

$L\sqrt{|\lambda|/k} = n\pi \text{ , } n=1,2,3\ldots$

$\lambda_n = k\left(n\pi/L\right)^2, \quad n=1,2,\ldots$

$\varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n=1,2,\ldots$

When $\lambda = \lambda_n$: $T'(t) + \lambda_n T(t) = 0$

$$T_n(t) = c_3 \exp(-\lambda_n t) = c_3 \exp\left(-k\left(\frac{n\pi}{L}\right)^2 t\right), \quad n=1,2\ldots$$

$u = u_n(x,t) = \varphi_n(x) T_n(t) = b_n \exp\left(-k\left(\frac{n\pi}{L}\right)^2 t\right) \sin\left(\frac{n\pi x}{L}\right)$

solves PDE for any constant b_n

Superposition: $u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-k\left(\frac{n\pi}{L}\right)^2 t\right) \sin\left(\frac{n\pi x}{L}\right)$ solves PDE.

At $t=0$: $u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

(a) $u(x,0) = 6\sin\left(\frac{9\pi x}{L}\right)$

Find constants b_n such that $6\sin\left(\frac{9\pi x}{L}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

Can do this by inspection, but instead use orthogonality:

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n \\ L/2, & m = n \end{cases}$$ (see, e.g. problem 5).

$$6\sin\left(\frac{9\pi x}{L}\right) \stackrel{?}{=} \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\int_0^L 6\sin\left(\frac{9\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \int_0^L \left(\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\right)\sin\left(\frac{m\pi x}{L}\right) dx$$

$$6\int_0^L \sin\left(\frac{9\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \sum_{n=1}^{\infty} b_n \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx}_{\text{only nonzero if } n=m}$$

$$= b_m \cdot L/2$$

$$b_m \cdot L/2 = 6\int_0^L \sin\left(\frac{9\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & m \neq 9 \\ 6L/2, & m = 9 \end{cases}$$

So $b_m = \begin{cases} 0, & m \neq 9 \\ 6, & m = 9 \end{cases}$

Full solution: $u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-k\left(\frac{n\pi}{L}\right)^2 t\right)\sin\left(\frac{n\pi x}{L}\right) = 6\exp\left(-k\left(\frac{6\pi}{L}\right)^2 t\right)\sin\left(\frac{6\pi x}{L}\right)$

(c) $u(x,0) = 2\cos\left(\frac{3\pi x}{L}\right)$

Find b_n such that $2\cos\left(\frac{3\pi x}{L}\right) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

$$2\int_0^L \sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{3\pi x}{L}\right)dx = \sum_{n=1}^{\infty} b_n \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$= b_m \cdot L/2$$

$$b_m = \frac{4}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{3\pi x}{L}\right)dx$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-k\left(\frac{n\pi}{L}\right)^2 t\right)\sin\left(\frac{n\pi x}{L}\right), \text{ where } b_n = \frac{4}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{3\pi x}{L}\right)dx$$

<u>13.3.5</u> $\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$

Then $\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{1}{2}\int_0^L \cos\left(\frac{(n-m)\pi x}{L}\right)dx - \frac{1}{2}\int_0^L \cos\left(\frac{(n+m)\pi x}{L}\right)dx$

If $m \neq n$: $\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{1}{2}\cdot\frac{L}{\pi(n-m)}\sin\left(\frac{(n-m)\pi x}{L}\right)\Big|_0^L - \frac{1}{2}\cdot\frac{L}{\pi(n+m)}\sin\left(\frac{(n+m)\pi x}{L}\right)\Big|_0^L$

$$= \frac{L}{2\pi(n-m)}(0-0) - \frac{L}{2\pi(n+m)}(0-0) = 0$$

If $m = n$: $\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{1}{2}\int_0^L 1\,dx - \frac{1}{2}\int_0^L \cos\left(\frac{(n+m)\pi x}{L}\right)dx$

$$= L/2 - \frac{L}{2\pi(n+m)}\sin\left(\frac{(n+m)\pi x}{L}\right)\Big|_0^L$$

$$= L/2$$

So $\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0, & m \neq n \\ L/2, & m = n \end{cases}$