MATH 3150 HW #2 Solutions

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**12.4.1** (a) Heat equation: $cp \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q$

Equilibrium $\Rightarrow \frac{\partial u}{\partial t} = 0$, heat equation becomes $\frac{d^2 u}{dx^2} = -Q/K_0$.

$Q=0 \Rightarrow \quad \frac{d^2 u}{dx^2} = 0, \quad u(0)=0, \quad u(L)=T$

$$\frac{du}{dx} = c_1$$

$$u(x) = c_1 x + c_2$$

$u(0)=0 \Rightarrow 0 = c_1 \cdot 0 + c_2$
$$c_2 = 0$$

$u(L)=T \Rightarrow \quad T = c_1 L + c_2$
$$T = c_1 L \rightarrow c_1 = T/L$$

$$u(x) = \frac{Tx}{L}$$

(b) $Q=0, u(0)=T, u(L)=0 \Rightarrow \frac{d^2 u}{dx^2} = 0, \quad u(0)=T \quad u(L)=0$

$$u(x) = c_1 x + c_2$$
$u(0)=T \Rightarrow c_2 = T$
$u(L)=0 \Rightarrow 0 = c_1 L + T \rightarrow c_1 = -T/L$

$$u(x) = T - \frac{Tx}{L}$$

(c) $Q=0, \frac{\partial u}{\partial x}(0)=0, u(L)=T \Rightarrow \frac{d^2 u}{dx^2}=0, \quad u'(0)=0, \quad u(L)=T$

$$u(x) = c_1 x + c_2$$

$u'(0)=0 \Rightarrow c_1 = 0$
$u(L)=T \Rightarrow c_2 = T$

$$u(x) = T$$

(d) $Q=0$, $u(0)=T$, ~~$KA$~~ $\frac{\partial u}{\partial x}(L)=\alpha$ $\implies$ $\frac{d^2u}{dx^2}=0$, $u(0)=T$, $u'(L)=\alpha$ ②

$$u(x)=c_1 x + c_2$$
$$u(0)=T \implies c_2 = T$$
$$u'(L)=\alpha \implies \alpha = c_1$$

$$u(x) = \alpha x + T$$

(e) $\frac{Q}{K_0}=1$, $u(0)=T_1$, $u(L)=T_2$ $\implies$ $\frac{d^2u}{dx^2}=-1$, $u(0)=T_1$, $u(L)=T_2$

$$u(x) = -\frac{x^2}{2} + c_1 x + c_2$$
$$u(0)=T_1 \implies T_1 = -0 + c_1 \cdot 0 + c_2 \longrightarrow c_2 = T_1$$
$$u(L)=T_2 \implies T_2 = -\frac{L^2}{2} + c_1 L + c_2$$

$$c_1 = \frac{1}{L}\left[T_2 - T_1 + \frac{L^2}{2}\right]$$

$$u(x) = -\frac{x^2}{2} + \frac{1}{L}\left[T_2 - T_1 + \frac{L^2}{2}\right] + T_1$$

<u>12.4.5</u> Equilibrium with no sources: $\frac{d^2u}{dx^2}=0$   $u(L)=T$, $T$ is unknown.

Known temperature @ $x=0 \rightarrow u(0)=T_0$ (known)

known heat flow @ $x=0 \rightarrow -K_0 \frac{du}{dx}(0) = \phi_0$ (known)

$$u''=0, \quad u(0)=T_0, \quad -K_0 u'(0)=\phi_0$$

$$u(x)=c_1 x + c_2$$
$$u(0)=T_0 \implies c_2 = T_0$$
$$-K_0 u'(0)=\phi_0 \implies -K_0 c_1 = \phi_0 \rightarrow c_1 = -\phi_0/K_0$$

$$u(x)= T_0 - \frac{\phi_0}{K_0} x$$

$$u(L)=T \implies \underline{T = T_0 - \frac{\phi_0 L}{K_0}}$$

12.4.7 (a) Equilibrium $\Rightarrow \frac{\partial u}{\partial t} = 0$

$$0 = \frac{d^2 u}{dx^2} + 1, \quad u(x,0) = f(x)$$

$$u'(0) = 1, \quad u'(L) = \beta.$$

$$u(x) = -\frac{x^2}{2} + c_1 x + c_2$$

$$u'(0) = 1 \Rightarrow 1 = 0 + c_1 \Rightarrow c_1 = 1$$

$$u'(L) = \beta \Rightarrow \beta = -L + c_1 = -L + 1, \quad \underline{\beta = 1 - L}$$

$$u(x) = -\frac{x^2}{2} + x + c_2$$

This value of $\beta$ (flux of heat at $x=L$) balances energy source terms and fluxes at boundaries.

Conservation of energy: at steady-state, energy in system should be conserved.

energy @ $t = 0$ : $\int_0^L u(x,0)\,dx = \int_0^L f(x)\,dx$  ($c\rho = 1$ from equation)

at steady-state: $\int_0^L u(x)\,dx = \int_0^L \left( -\frac{x^2}{2} + x + c_2 \right) dx$

$$= -\frac{L^3}{6} + \frac{L^2}{2} + c_2 L = \int_0^L f(x)\,dx$$

$$c_2 = \frac{1}{L}\left[ \int_0^L f(x)\,dx + \frac{L^3}{6} - \frac{L^2}{2} \right]$$

$$u(x) = -\frac{x^2}{2} + x + \frac{1}{L}\left( \int_0^L f(x)\,dx + \frac{L^3}{6} - \frac{L^2}{2} \right)$$

(b) Equilibrium $\rightarrow \frac{\partial u}{\partial t} = 0$

$$\frac{d^2 u}{dx^2} = 0 \rightarrow u(x) = c_1 + c_2 x$$

$$u'(0) = 1 \Rightarrow c_2 = 1 \left. \right\}$$
$$u'(L) = \beta \Rightarrow c_2 = \beta \left. \right\} \beta = 1$$

There are no sources in this PDE. $\beta = 1$ ensures that heat flux out equals that in → i.e. conservation of energy.

To determine constant $C_1$: $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$

$$\int_0^L \frac{\partial u}{\partial t}\, dx = \int_0^L \frac{\partial^2 u}{\partial x^2}\, dx$$

$$\frac{\partial}{\partial t}\left(\int_0^L u\, dx\right) = \left.\frac{\partial u}{\partial x}\right|_0^L = \frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t) = \beta - 1 = 0$$

So $\int_0^L u\, dx$ is constant for all time

at $t=0$: $\int_0^L u(x,0)\, dx = \int_0^L f(x)\, dx$

at equilibrium: $\int_0^L u(x)\, dx = \int_0^L (C_1 + x)\, dx = C_1 L + \dfrac{L^2}{2}$

$$C_1 L + \frac{L^2}{2} = \int_0^L f(x)\, dx$$

$$C_1 = \frac{1}{L}\left(\int_0^L f(x)\, dx - \frac{L^2}{2}\right) \quad \longrightarrow \quad u(x) = x + \frac{1}{L}\left(\int_0^L f(x)\, dx - \frac{L^2}{2}\right)$$

(c) $\dfrac{\partial u}{\partial t} = 0 \longrightarrow \dfrac{d^2 u}{dx^2} + x - \beta = 0$

$$u'' = \beta - x$$

$$u' = \beta x - \frac{x^2}{2} + c_1$$

$$u = \frac{\beta}{2}x^2 - \frac{x^3}{6} + c_1 x + c_2$$

$$u'(x) = \beta x - \frac{x^2}{2} + c_1 \longrightarrow u'(0) = 0$$

$$\beta \cdot 0 - \frac{0^2}{2} + c_1 = 0$$

$$c_1 = 0$$

$$u'(L) = 0$$

$$\beta L - \frac{L^2}{2} = 0$$

$$\beta = \frac{L}{2} \quad \text{This value of } \beta \text{ ensures that the net energy imparted to the system from the source term is 0.}$$

$u(x) = \frac{\beta}{2}x^2 - \frac{x^3}{6} + C_2$

To determine $C_2$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x - \beta$$

$$\frac{\partial}{\partial t}\int_0^L u \, dx = \int_0^L \frac{\partial^2 u}{\partial x^2}\, dx + \int_0^L (x - \beta)\, dx$$

$$= \frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t) + \frac{x^2}{2}\Big|_0^L - \beta L$$

$$= 0 - 0 + \frac{L^2}{2} - \frac{L}{2}\cdot L = 0$$

Again, $\int_0^L u(x,t)\,dx$ is constant in $t$.

At $t=0$: $\int_0^L u(x,0)\,dx = \int_0^L f(x)\,dx$

At equilibrium: $\int_0^L \frac{\beta}{2}x^2 - \frac{x^3}{6} + C_2 \, dx = \frac{1}{6}\cdot\frac{L}{2}\cdot L^3 - \frac{L^4}{24} + C_2 L = \frac{L^4}{24} + C_2 L$

$$C_2 = \frac{1}{L}\left(\int_0^L f(x)\,dx - \frac{L^4}{24}\right)$$

$$u(x) = \frac{L}{4}x^2 - \frac{x^3}{6} + \frac{1}{L}\left(\int_0^L f(x)\,dx - \frac{L^4}{24}\right)$$

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**13.2.3** $\quad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \alpha(x,t)u + \beta(x,t)$

Let $f = \beta(x,t)$

Let $L(u) = \frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} - \alpha(x,t)u$

Then the PDE is of the form $\quad Lu = f$.

Is $L$ linear? Let $u_1$ and $u_2$ be functions, $C_1$ and $C_2$ be constants:

$$L(C_1 u_1 + C_2 u_2) = \frac{\partial}{\partial t}(C_1 u_1 + C_2 u_2) - k\frac{\partial^2}{\partial x^2}(C_1 u_1 + C_2 u_2) - \alpha(x,t)[C_1 u_1 + C_2 u_2]$$

$$= C_1\frac{\partial u_1}{\partial t} - k\frac{\partial^2 u_1}{\partial x^2}\cdot C_1 - \alpha(x,t)C_1\cdot u_1 + C_2\frac{\partial u_2}{\partial t} - C_2 k\frac{\partial^2 u_2}{\partial x^2} - C_2\alpha(x,t)u_2$$

$$= c_1 L(u_1) + c_2 L(u_2)$$

→ so $L$ is linear, hence the PDE is linear.

If $\beta(x,t)=0$, then PDE is of the form $Lu=0$, and is thus homogeneous.

**3.2.4** (a) $L(u_p)=f$, $L(u_1)=0$, $L(u_2)=0$, $L$ is linear.

Then $L(u_p + c_1 u_1 + c_2 u_2) = L(u_p) + c_1 L(u_1) + c_2 L(u_2)$
$$= f + 0 + 0 = f$$

Hence $u_p + c_1 u_1 + c_2 u_2$ is a particular solution.

(b) $L(u) = f_1 + f_2$, and $L(u_1)=f_1$, $L(u_2)=f_2$, $L$ is linear

Then $L(u_1 + u_2) = L(u_1) + L(u_2) = f_1 + f_2$

So $u_1 + u_2$ is a particular solution for $L(u) = f_1 + f_2$.