

This test is:

- closed-book
- closed-notes
- no-calculator
- 120 minutes

Indicate your answers clearly, and show your work. Partial credit will be awarded based on work shown. Full credit will not be awarded without some work shown.

Fun fact of life: if your work is not legible, I will not be able to read it. The ramifications of this outcome should be clear.

There are 4 questions with multiple parts; each question is worth 25 points.

All pages are one-sided. If on any problem you require more space, use the back of the page.

DO NOT TURN THIS PAGE UNTIL DIRECTED TO BEGIN

1. (25 pts) Compute the solution $u(x, t)$ to the following one-dimensional heat equation:

$$u_t = u_{xx}, \quad 0 < x < L, \quad t > 0$$

subject to the initial and boundary conditions

$$\begin{aligned} u(x, 0) &= x^2(x - L)^2, \\ \frac{\partial u}{\partial x}(0, t) &= 0, \\ \frac{\partial u}{\partial x}(L, t) &= 0 \end{aligned}$$

Show all work. Your solution must be written down in terms of explicit, computable expressions or integrals, but you need not compute the values of these integrals.

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2. (25 pts total) This problem concerns eigenvalue problems and Fourier Series.

a.) (15 pts) Solve the following eigenvalue problem: find all eigenvalues λ and eigenfunctions $\phi(x)$. You must show all work, including exhausting all possible values of λ .

$$\begin{aligned}\phi''(x) + \lambda\phi(x) &= 0, & 0 < x < L \\ \phi(0) &= 0, & \phi(L) &= 0\end{aligned}$$

b.) (10 pts) Compute the Fourier Series coefficients on the interval $[-L, L]$ for the function

$$f(x) = \begin{cases} -1, & x \geq 0 \\ 1, & x < 0 \end{cases}$$

3. (25 pts total) This problem is concerned with the Fourier Transform and its properties.

a.) (8 pts) Show, using only the definition of the Fourier Transform, that the Fourier Transform of $f(x - 3)$ equals $F(\omega)e^{i3\omega}$.

b.) (7 pts) Compute the Fourier Transform of $g(x) = \frac{d}{dx}(\exp(-x^2))$.

c.) (7 pts) Compute the Fourier Transform of $h(x) = x^2 \exp(-x^2)$.

d.) (3 pts) Let $f(x) = \exp(-x^2)$. Compute an explicit expression for $f * f$.

4. (25 pts total) Solve the following PDE:

$$u_t = u_{xx} + u_x, \quad -\infty < x < \infty, \quad t > 0$$

subject to the initial condition

$$u(x, 0) = \exp(-|x|)$$

Show all work. Your solution must be written down in terms of explicit, computable expressions or integrals, but you need not compute these integrals.

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