Department of Mathematics, University of Utah

Orthogonal polynomials/Spectral methods for PDEs
MATH 5750/6880 – Section 002 – Fall 2018
Midterm project
Approximations with polynomials

Due October 4, 2018

In all problems below, $\{p_n\}_{n\in\mathbb{N}_0}$ is a collection of L^2_w-orthonormal polynomials where w is a non-negative weight function on $\mathbb{R}$. The three-term recurrence coefficients for this family are $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}_0}$. P_n is the space of univariate polynomials of degree n or less.

1. Assume that w has compact support on $[a,b] \subset \mathbb{R}$. This problem concerns computation of quadrature rules with *prescribed* nodes outside the open interval (a,b). For this purpose, a quadrature rule is called *optimal* if its order of accuracy is as large as possible. For example, an n-point Gaussian quadrature rules, which has no *a priori* prescribed nodes, is optimal, having order of accuracy $2n-1$. We assume below that $1 \le q \le n$.

a.) Suppose that $\{x_1,\ldots,x_q\}$ is a set of points *outside* (a,b). Show that an n-point rule with q prescribed nodes at $\{x_1,\ldots,x_q\}$ cannot have order of accuracy $2n-q$.

b.) Under the conditions in part a, show that an order-$(2n-q-1)$ accurate rule is given by an interpolatory quadrature rule, whose nodes are $\{x_1,\ldots x_q\}$ unioned with the $(n-q)$-point Gaussian quadrature nodes for the weight function

$$v(x) \coloneqq w(x)\prod_{j=1}^{q}|x-x_j|.$$

(Note that since the x_j are outside (a,b), then v is just w multiplied by a polynomial on $[a,b]$. The absolute value bars only serve to ensure that v has a non-negative value on $[a,b]$.)

c.) Let $\alpha,\beta > -1$ be fixed, arbitrary real numbers, and consider the family of (α,β)-parameter Jacobi polynomials on $[-1,1]$. A (Jacobi-)*Gauss-Radau* quadrature rule, with prescribed node at $x=+1$, is an n-point quadrature rule that is optimal. Explain how to compute the nodes of the (α,β) Gauss-Radau quadrature rule in terms of (standard) Gaussian quadrature rules of other Jacobi polynomial families.

d.) Explain how to compute the nodes of the (α,β) Gauss-Radau quadrature rule with prescribed node at $x=-1$ in terms of (standard) Gaussian quadrature rules of other Jacobi polynomial families.

e.) A (Jacobi-)*Gauss-Lobatto* quadrature rule is an optimal quadrature rule with 2 prescribed nodes at $x=\pm 1$. Explain how to compute the nodes of the (α,β) Gauss-Lobatto quadrature rule in terms of (standard) Gaussian quadrature rules of other Jacobi polynomial families.

2. This problem concerns the Gauss-Radau rules considered in problem # 1, but in a more general context. Here, assume w is any weight function.

a.) Define the ratio of two successive polynomials in the L^2_w-orthonormal sequence,

$$r_n(x) := \frac{p_n(x)}{p_{n-1}(x)}, \qquad\qquad n \geq 1.$$

Show that $r_n'(x) > 0$ for all $x \in \mathbb{R}\backslash p_{n-1}^{-1}(0)$.

b.) Prove that any level set of r_n has exactly n points. I.e., show that $\left|r_n^{-1}(c)\right| = n$ for any $c \in \mathbb{R}$, where $|\cdot|$ denotes the number of elements in a subset of $\mathbb{R}$.

c.) Fix $a \in \mathbb{R}$. Show that, if $a \notin p_{n-1}^{-1}(0)$, then $a \in r_n^{-1}\left(r_n(a)\right)$.

d.) Fix $c \in \mathbb{R}$. Prove that the n nodes $\{x_1, \ldots, x_n\} := r_n^{-1}(c)$, along with the n weights defined by

$$w_j := \left(\sum_{k=0}^{n-1} p_k^2(x_j)\right)^{-1}, \qquad\qquad 1 \leq j \leq n,$$

form an n-point quadrature rule that is exact for polynomials of degree at least $2n-2$. For which values of c does the quadrature rule have the optimal order of $2n-1$?

e.) Fix $a \in \mathbb{R}\backslash p_{n-1}^{-1}(0)$. Describe an algorithm that computes an n-point quadrature rule, with one node prescribed at $x = a$, with order of accuracy at least $2n-2$. Your algorithm should use spectral quantities of a Jacobi matrix, similar to how standard Gauss quadrature rules are computed, but this Jacobi matrix need not be the one associated to w. You do not need to implement such an algorithm.

3. This problem studies approximations of derivatives. Consider the class of functions f_q from the previous assignment:

$$f_0(x) = \begin{cases} 0, & x < 0 \text{ and } x > 1 \\ 1, & 0 \leq x \leq 1 \end{cases}$$

$$f_q(x) = \frac{1}{C_q}\int_{-1}^{x} f_{q-1}(s)\,\mathrm{d}s, \qquad C_q = \int_{-1}^{1} f_{q-1}(s)\,\mathrm{d}s, \quad q \geq 1$$

For $x \in [-1,1]$ with the Legendre weight, numerically compute algebraic rates of convergence in N for N-point Gauss quadrature interpolants for $f_q^{(j)}$, $0 \leq j \leq q \leq 4$, where $f_q^{(j)}$ is the jth derivative of f_q. Discuss how differentiation affects rates of convergence.

4. For $f \in L^2_w$, the best L^2_w degree-N approximation to f is the Fourier projection f_N:

$$f_N(x) \coloneqq \sum_{n=0}^{N} \widehat{f}_n p_n(x), \qquad \widehat{f}_n \coloneqq \langle f, p_n \rangle_w .$$

A second approximation is the $(N+1)$-point Gauss quadrature interpolant $I_N f$:

$$I_N f \coloneqq \sum_{j=0}^{N} \widetilde{f}_n p_n(x), \qquad I_N f(x_j) = f(x_j), \qquad \{x_j\}_{j=1}^{N+1} = p_{N+1}^{-1}(0).$$

The difference $f_N - I_N f$ is called *aliasing error*. For w the Legendre weight function on $[-1, 1]$, compute L^2_w and L^∞ norms for the aliasing error for the functions $f^{(q)}$ above, for $q = 0, 1, 2$ and various N. Compare norms of the aliasing error to those of the Fourier projection error, $f - f_N$. In particular, analyze convergence rates of the aliasing error to 0 compared to convergence rates of $f_N - f$ and discuss how this depends on q. For this example (this choice of w and functions) does interpolation negatively affect rates of convergence for Fourier projections?