Department of Mathematics, University of Utah

Orthogonal polynomials/Spectral methods for PDEs
MATH 5750/6880 – Section 002 – Fall 2018
Homework 4
Solutions to ordinary differential equations

Due November 20, 2018

In the problems below, $\{p_n\}_{n\in\mathbb{N}_0}$ is a collection of L^2_w-orthonormal polynomials where w is a non-negative weight function on $\mathbb{R}$. The three-term recurrence coefficients for this family are $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}_0}$.

1. Consider the ordinary differential equation

$$\begin{aligned} -u''(x) + u(x) &= f(x), & x \in (-1,1) \\ u(-1) = u(1) &= 0, & \end{aligned}$$

Choose $f(x)$ so that the solution is $u(x) = (1-x^2)\sin(10x)$. Consider the following ("Legendre"-)Galerkin procedure for computing discrete solutions: find $u_N \in P_{N,0}$ satisfying

$$\int_{-1}^{1} u_N'(x) v'(x)\,\mathrm{d}x + \int_{-1}^{1} u_N(x) v(x)\,\mathrm{d}x = \int_{-1}^{1} f(x) v(x)\,\mathrm{d}x, \qquad v \in P_{N,0},$$

where the polynomial space $P_{N,0}$ is defined as:

$$P_{N,0} = \left\{ p \;\middle|\; \deg p \leq N, \;\; p(-1) = p(1) = 0 \right\}.$$

a.) Use a basis v_n, $n \geq 1$, for $P_{N,0}$ defined as

$$v_n(x) = p_{n+1}(x) + c_n p_0 + d_n p_1(x),$$

where p_n are the orthonormal Legendre polynomials, and c_n and d_n are chosen to ensure that $v_n \in P_{N,0}$. Compute the $(N-1)\times(N-1)$ "stiffness" matrix S^v defined as

$$S^v_{j,k} = \int_{-1}^{1} v_j'(x) v_k'(x)\,\mathrm{d}x.$$

Plot the sparsity pattern for this definition of S.

b.) Repeat the formulation above, using a *different* basis w_n for $P_{N,0}$,

$$w_n(x) = p_{n+1}(x) + e_n p_n(x) + f_n p_{n-1}(x),$$

where again e_n and f_n are chosen to ensure that $w_n \in P_{N,0}$. For this basis, plot the sparsity pattern for S^w.

c.) *Banded* matrices can be efficiently inverted. (Precisely, if A is $N \times N$ and banded, then $A^{-1}x$ can be evaluated for an arbitrary vector x with $O(N)$ cost.) Which of the two methods above would you prefer for implementation purposes?

d.) Numerically verify that accuracy of the Legendre-Galerkin approximation u_N is exponential in N.

2. Consider the differential equation,

$$\begin{aligned} -u''(x) - u(1-u) &= f(x), & x \in (-1,1) \\ u(-1) = u(1) &= 0, \end{aligned}$$

Use a collocation method based on N Legendre-Gauss-Lobatto points to solve this equation. Demonstrate exponential convergence as a function of N for smooth u by choosing an appropriate function for f so you can identify an exact solution.
Optional: Use a Legendre-Galerkin method on $P_{N,0}$ and demonstrate exponential convergence in N.

3. Consider the biharmonic differential equation

$$\begin{aligned} u^{(4)} - u_{xx} + u &= f(x), & x \in (-1,1) \\ u(-1) = u(1) &= 0, & u'(-1) = u'(1) = 0. \end{aligned}$$

Consider a Galerkin procedure to find a solution u from the polynomial space

$$P_{N,0,0} = \left\{ p \;\middle|\; \deg p \leq N, \;\; p(-1) = p(1) = p'(-1) = p'(1) = 0 \right\},$$

which is a subspace of $H_0^2([-1,1])$. Derive an appropriate bilinear form for the differential equation using this space. Numerically implement a Galerkin solver for this equation, using basis functions v_n for $P_{N,0,0}$ defined as

$$v_n(x) = \sum_{j=n-1}^{n+3} c_{n,j} p_j(x), \qquad n \geq 1.$$

where $c_{n,j}$ are chosen so that $v_n \in P_{N,0,0}$. Demonstrate exponential accuracy of your solver as a function of N.