Department of Mathematics, University of Utah

Orthogonal polynomials/Spectral methods for PDEs

MATH 5750/6880 – Section 002 – Fall 2018

Homework 2

Quadrature and approximations

Due September 20, 2018

In all problems below, $\{p_n\}_{n \in \mathbb{N}_0}$ is a collection of L^2_w-orthonormal polynomials where w is a non-negative weight function on $\mathbb{R}$. The three-term recurrence coefficients for this family are $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}_0}$. P_n is the space of univariate polynomials of degree n or less.

1. Consider the weight function with support on $[-1, 1]$ given by

$$w(x) = \frac{1}{\pi\sqrt{1-x^2}}.$$

a.) Show that the sequence of polynomials $\{T_n\}_{n\geq 0}$ defined by $T_n(x) \coloneqq \cos\left(n \arccos x\right)$, are L^2_w orthogonal, and compute the associated normalizing constants that would define an orthonormal sequence $\{p_n\}_{n\geq 0}$ in terms of $\{T_n\}_{n\geq 0}$. The polynomials T_n are called *Chebyshev polynomials.*

b.) Compute an explicit formula for the zeros of the Chebyshev polynomials, hence for the nodes of a w-Gaussian quadrature rule.

2. Fix $n \in \mathbb{N}$.

a.) Let $x \in \mathbb{R}$ be fixed but arbitrary. Consider the linear functional δ_x, corresponding to point-evaluation at x:

$$\delta_x : C(\mathbb{R}) \to \mathbb{R}, \qquad\qquad \delta_x(f) = f(x),$$

where $C(\mathbb{R})$ is the space of continuous functions on $\mathbb{R}$. Consider the space P_{n-1}, and note that $P_{n-1} \subset C(\mathbb{R}) \cap L^2_w$. Compute an explicit expression, in terms of the orthonormal family p_n, for the operator norm of δ_x induced by the norm on $P_{n-1} \subset L^2_w$:

$$\|\delta_x\| \coloneqq \sup_{f \in P_{n-1}\setminus\{0\}} \frac{|\delta_x(f)|}{\|f\|_{L^2_w}}.$$

b.) Let $x_1, \ldots, x_n$ be the nodes of the n-point Gaussian quadrature rule. Let V be the associated $n \times n$ Vandermonde-like matrix with entries

$$(V)_{j,k} = p_{k-1}(x_j), \qquad\qquad 1 \leq j, k \leq n.$$

Show that the singular values of V are equal to $\|\delta_{x_j}\|$, $j = 1, \ldots, n$.

---

3. Let w be the constant function on $[-1, 1]$ and zero elsewhere. Consider the function,

$$g_n(x) \coloneqq x^n + x^2,$$

over the interval $[-1, 1]$. Use a $2n$-point Gauss quadrature rule to compute an interpolant in P_{2n-1} in the basis $\{p_n\}$,

$$\widetilde{g}_n(x) = \sum_{j=0}^{2n-1} c_j p_j(x).$$

We seek to compute new coefficients d_j such that

$$\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{g}_n(x) = \sum_{j=0}^{2n-1} d_j p_j(x).$$

Consider the following two (mathematically equivalent) ways to compute d_j:

- Use coefficients $C_{n,j}$ defined as

$$p_n'(x) = \sum_{j=0}^{n-1} C_{n,j} p_j(x),$$

  to *directly* transform c_j to d_j.

- Transform $\widetilde{g}_n$ to a monomial representation, differentiate with the power rule, and then transform back to the basis p_j:

$$\sum_{j=0}^{2n-1} c_j p_j(x) \quad \longrightarrow \quad \sum_{j=0}^{2n-1} m_j x^j \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}x}} \quad \sum_{j=1}^{2n-1} j m_j x^{j-1} \quad \longrightarrow \quad \sum_{j=0}^{2n-1} d_j p_j(x).$$

There are tools in `pyopoly` that can help in computing $C_{n,j}$ and connecting to monomials and back, see the methods `canonical_connection`, `canonical_connection_inverse`, and `derivative_expansion` in the `OrthogonalPolynomialBasis1D` class. Implement both of the above procedures. For $n = 2, 3, 4, \ldots, 50$, generate plots with numerical results that demonstrate (a) that $\widetilde{g}_n$ accurately approximates g_n to considerable precision both in the L^2_w and L^∞ norms, (b) that when computing derivatives with large n, using the $C_{n,j}$ coefficients is more accurate in both L^2_w and L^∞ norms than going through monomials. For large n, which step of the monomial process introduces errors?

4. Consider the functions f_q, $q \geq 0$, defined by

$$f_0(x) = \begin{cases} 0, & x < 0 \text{ and } x > 1 \\ 1, & 0 \leq x \leq 1 \end{cases}$$

$$f_q(x) = \frac{1}{C_q} \int_{-1}^{x} f_{q-1}(s)\,\mathrm{d}s, \qquad C_q = \int_{-1}^{1} f_{q-1}(s)\,\mathrm{d}s, \quad q \geq 1$$

Consider approximating these functions in L^2_w, where w is the constant weight function on $[-1, 1]$. For each of $q = 0, 1, 2, 3$, compute N-point Gauss qaudrature interpolants for these functions for $N = 1, \ldots, 200$, along with numerically-computed L^2_w and L^∞ errors. How do the errors of each of these functions decay as N is increased? Can you make a conjecture about what property of these functions governs the rate of convergence?