Department of Mathematics, University of Utah

Orthogonal polynomials/Spectral methods for PDEs

MATH 5750/6880 – Section 002 – Fall 2018

Homework 1

Orthogonal polynomials

Due September 6, 2018

In all problems below, $\{p_n\}_{n\in\mathbb{N}_0}$ is a collection of L^2_w-orthonormal polynomials where w is a non-negative weight function on $\mathbb{R}$. The three-term recurrence coefficients for this family are $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}_0}$.

1. Let p_n have (positive) leading coefficient $\kappa_n > 0$, i.e.,

$$p_n(x) = \kappa_n x^n + \cdots .$$

Define $P_n(x) = p_n(x)/\kappa_n$. The family $\{P_n\}_{n\in\mathbb{N}_0}$ are called a *monic* orthogonal polynomial sequence.

a.) Compute κ_n explicitly in terms of the recurrence coefficients for p_n.

b.) The P_n satisfy a three-term recurrence formula

$$xP_n(x) = A_{n+1}P_{n+1}(x) + B_{n+1}P_n(x) + C_{n+1}P_{n-1}(x), \quad n \in \mathbb{N}_0.$$

Derive expressions for the (A_n, B_n, C_N) in terms of the recurrence for the original family.

2. Define a new weight function

$$\omega(x) = w(bx + a), \qquad b > 0, \quad a \in \mathbb{R}.$$

Let $\{\pi_n\}_{n\in\mathbb{N}_0}$ be an orthonormal polynomial family in L^2_ω. Compute the recurrence coefficients for π_n in terms of those for p_n.

3. Suppose w is an even function. Show that

a.) $a_n = 0$ for all n.

b.) p_{2k} is an even function for all $k \in \mathbb{N}_0$.

c.) p_{2k+1} is an odd function for all $k \in \mathbb{N}_0$.

4. Assume $w(x)$ has compact support on $[a, b] \subset \mathbb{R}$ for some $a, b \in \mathbb{R}$. I.e., $w(x) = 0$ for $x \notin [a, b]$. Let $x_0 < a$ be arbitrary and define $\omega(x) = (x - x_0)w(x)$. Let $\{\pi_n\}_{n \in \mathbb{N}_0}$ be the family of L^2_ω-orthonormal polynomials. Show that there exist constants u_n, v_n, such that

$$(x - x_0)\pi_n(x) = u_n p_{n+1}(x) + v_n p_n(x), \qquad n \geq 0.$$

(This is a specialization of a *Christoffel Theorem.*) What does this result tell you about the connection coefficients between $\{p_n\}$ and $\{\pi_n\}$?