Mesh Quality for Three-dimensional Finite Element Solutions on Anisotropic Meshes

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Abstract

The concept of solution-based mesh quality is driven by the need not only to consider whether a good solution can be produced on a given mesh, but also whether a mesh can be considered appropriate for a given solution. The aim of this paper is to look at different ways in which such a mesh can be produced for a model hyperbolic problem, which may be regarded as typical of flow problems with directional solutions. Given an error estimate, standard h-refinement schemes can reduce this error to below user-specified levels by increasing the resolution where the error is large, however in three dimensions such an approach can lead to excessively large meshes. In particular, for problems containing strongly directional features it may be possible to reduce the error by refining the mesh in a directional manner, and thus with greater efficiency than uniform h-refinement. The approach adopted is to consider an r-refinement scheme combined with an existing isotropic h-refinement approach, with the node movement driven by a selection of solution-based mesh quality measures.

1 Introduction

Mesh quality has in the past been considered purely in terms of geometry [4], based on analysis of elliptic problems with isotropic solutions. The early measures of geometric mesh quality were the minimum angle condition for triangles of Zlamal [17] and then, more precisely, the maximum angle condition of Babuska and Aziz [1]. However the desire to solve complex three-dimensional (3D) problems has led to the use of anisotropic meshes, for example to model boundary layers in Navier-Stokes simulations, that would not presently be feasible with an isotropic grid. In these cases the standard geometric measures are inadequate. The quality of the mesh depends on the relationship between the shape of an element and the numerical error resulting from the use of the given discretisation method to solve the problem being considered. Solution-based measures of mesh quality link the mesh geometry to the error in the approximate solution and are surveyed in [3] and [4].

At present, a widely used solution-based measure of mesh quality is the Hessian matrix, consisting of second derivatives of a scalar function, which is the solution for a scalar equation or one component of a system. This was initially based on 1D considerations of interpolation error estimates [13]. The eigen-decomposition of the matrix gives information about the principal directions of solution variation and their magnitude. The assumption is made that a mesh with size inversely proportional to the eigenvalue magnitudes in the respective directions will approximately equidistribute the interpolation error. This appears to have been first applied to 2D [13] and 3D [12] finite element

(FE) calculations by Peraire *et al.* In this case the Hessian information was used to drive a remeshing of the domain with an advancing front technique. Other work has included the use of local mesh operations to improve an existing grid [6].

At a more abstract level Berzins [2] has shown how the interpolation error estimate for linear tetrahedra may be written in terms of edge second derivatives; the L^2 norm of the interpolation error is estimated by considering the difference between the linear interpolant and a locally quadratic approximation to the exact solution on each tetrahedral element. This has then been used this to define a solution-based mesh quality measure with properties similar to the Hessian-based approach. Both these and many of the previous approaches to mesh quality are based on interpolation error estimates. In this paper alternative measures are also considered based on residual error estimates and on an estimate of the error in the FE solution. The problem considered here is not only whether a mesh can be quantified as appropriate for a given solution but also whether the measure can be used to drive the mesh into a more appropriate form, and hence increase the accuracy of the solution. Considerations of mesh alignment and an r-refinement scheme are used here to produce anisotropic meshes.

These new techniques are evaluated by using a simple linear hyperbolic model problem:

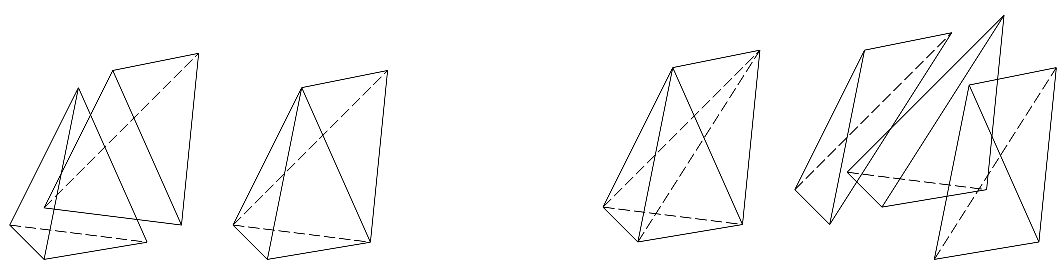
$$(\mathbf{a}\cdot\nabla)\,u = f \quad in\ \Omega : \{\mathbf{x}\in[0,1]^3\}, \tag{1}$$

$$u = \begin{cases} 1 - x/\delta & x < \delta \\ 0 & x \geq \delta \end{cases} \quad on\ \Gamma_{in} : \{\mathbf{x}\in\Gamma, \mathbf{x}\cdot\mathbf{n} < 0\}, \tag{2}$$

with a constant convection velocity $\mathbf{a} = (2,1,1)$ and $\delta = 0.01$ used throughout. The solution consists of a strongly directional layer that is advected through the domain.

2 Mesh alignment

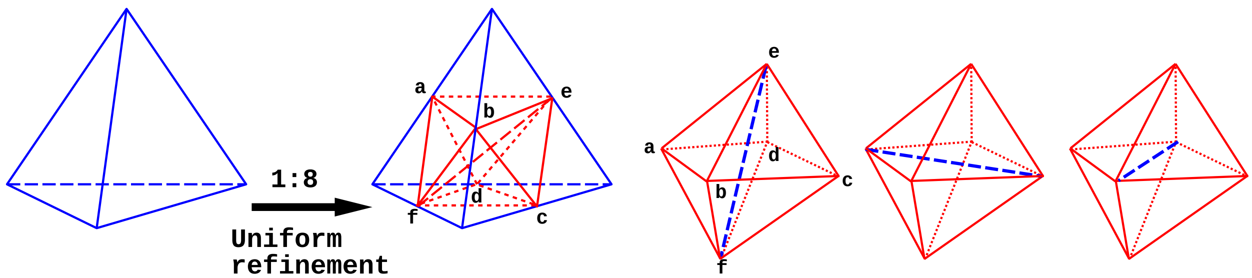
In order to produce anisotropic meshes for problems with directional solutions it may be necessary to align the mesh so as to improve its "quality" with respect to the solution. This issue has been considered by several researchers. Iliescu [8] shows that in 2D and 3D the solution accuracy for a linear hyperbolic problem can be increased by simple mesh reconnection to better align the mesh with the convection direction. This involves edge swapping in 2D, and a face swapping operation in 3D that is shown in Figure 1. Madden and Stynes [11] consider the *a priori* generation of aligned meshes for convection-diffusion equations and show that improved accuracy is obtained.



(a) Two initial tetrahedra — (b) Three aligned tetrahedra

Figure 1: 3D alignment operation for 2 tetrahedra

A further observation on mesh alignment can be made by considering the isotropic h-refinement of a tetrahedral element, which produces 8 *child* elements and is shown in Figure 2. There is a choice of three diagonals for the central volume and this can be chosen by either geometric criteria or



(a) Isotropic h-refinement (b) Diagonal choices for the central volume

Figure 2: Isotropic refinement of a tetrahedron

by some solution-based criteria. A simple experiment shows that by following Iliescu's reasoning and choosing this diagonal to be aligned with the convection direction one obtains an immediate reduction in the solution error, which is computed using a standard Streamline-Upwind Petrov-Galerkin (SUPG) FE method. The mesh is a uniform 9^3 discretisation of the domain formed by uniformly refining an initial 3^3 mesh twice with the diagonals chosen according to the two criteria. The L^1 errors for the model problem are 0.100 and 0.067 for the geometrical and aligned diagonal choices respectively, confirming the expected increase in accuracy.

The anisotropic *a posteriori* error estimates for elliptic problems of Kunert [10] introduces the concept of a matching function which measures the correspondence of the mesh with the solution. It is shown that a good alignment of the mesh to the solution is required to give sharp error estimates. Kunert has also shown that, under a number of assumptions, meshes produced by the Hessian approach lead to an appropriate matching function for the error estimate to be sharp. The simple experiment described previously suggests that a similar quality of alignment is also desirable in the hyperbolic case.

The mesh alignment techniques presented previously depend on *a priori* knowledge in order to specify an appropriate alignment for the mesh. In the following sections it will be shown that in some cases these choices can be made *a posteriori*, and thus adaptively, and more precisely quantified.

3 An hr-refinement scheme based on interpolation error

Motivated by Berzins' interpolation error estimate [2], which expresses the L^2 norm of the interpolation error in a tetrahedral cell as a combination of the second derivatives along the six edges d_s of a tetrahedral cell of volume V^k:

$$\int_k e_{lin}^2 \, d\Omega \;=\; \frac{6}{4} V^k \frac{2}{7!} \left(\left(\sum_{s=1}^{6} d_s \right)^2 - d_1 d_4 - d_2 d_5 - d_3 d_6 + \sum_{s=1}^{6} d_s^2 \right), \tag{3}$$

an r-refinement node movement scheme is proposed which attempts to equidistribute the edge second derivatives throughout the mesh. The node movement scheme is driven by using the edge second derivatives in a weighted average expression for the node position. A relaxation scheme is used in which the nodal positions are updated by

$$\mathbf{x}_i^{av} \;=\; \frac{\sum_{s \in \Omega_i} |d_s| \, \mathbf{x}_j}{\sum_{s \in \Omega_i} |d_s|}, \qquad \mathbf{x}_i \;\rightarrow\; (1-\gamma_i)\mathbf{x}_i + \gamma_i \mathbf{x}_i^{av}, \tag{4}$$

where s is the edge ij and Ω_i is the patch of elements connected to node i. γ_i is a safety factor at each node i that prevents the mesh from becoming tangled by restricting the node movement to be within one half of the perpendicular distance to the closest opposing tetrahedral face. In practice this procedure is implemented in a Jacobi fashion by first assembling all of the node position increments and then updating the entire mesh, and several sweeps are performed at each r-refinement stage. Note the similarity of equidistribution of the edge second derivatives with the Hessian approach, i.e. a tetrahedron with $|d_s|$ equidistributed over the edges should appear isotropic within the Hessian metric space.

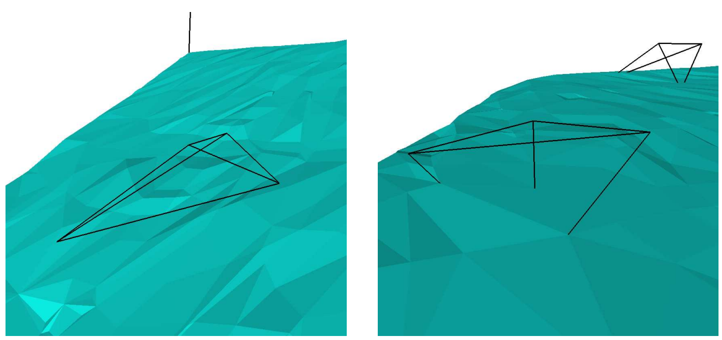
h-refinement		hr-refinement	
Np	$\|\|e\|\|_1$	Np	$\|\|e\|\|_1$
729	0.111	729	0.111
			0.099
			0.093
2761	0.070	2736	0.073
			0.068
			0.065
12026	0.045	11889	0.047
			0.043
			0.041
52871	0.027	57411	0.030
			0.028
			0.026

(a) Misaligned mesh

h-refinement		hr-refinement	
Np	$\|\|e\|\|_1$	Np	$\|\|e\|\|_1$
729	0.092	729	0.092
			0.088
			0.080
2637	0.059	2690	0.054
			0.051
			0.048
11182	0.035	11323	0.029
			0.028
			0.028
49225	0.021	48930	0.020
			0.018
			0.017

(a) Better aligned mesh

Table 1: Comparison of the L^1 norm of the error



(a) Low error (b) High error

Figure 3: Example tetrahedra relative to the $u = 0.5$ isosurface

Table 1 compares the L^1 norm of the error for the model hyperbolic problem for an isotropic h-refinement scheme and for a hr-refinement scheme using the node movement algorithm (4) prior to each h-refinement step. In this case two Jacobi sweeps were performed on the mesh with a fixed solution, then the solution was recomputed. This was repeated once, and then h-refinement was performed. The results indicate that while the node movement scheme can help to reduce the error in the problem the improvement is not great in this case, although a lower error was obtained on each grid at the end of the r-refinement stage. Table 1(b) shows that the results are improved by choosing a more aligned initial mesh (in the sense of Iliescu [8]) as expected. There are various parameters in the adaptivity algorithm that can be tuned, such as the number of sweeps and the number of times the solution is recomputed during r-refinement, and improved results can be obtained by repeated experimentation. In practice the r-refinement scheme never does worse than h-refinement alone, and in some cases much greater improvements can be obtained.

It is important to note that this scheme is independent of the underlying problem being solved, and is based purely on estimates of interpolation error. This may be advantageous in the sense that it can be applied to a wide range of problems. In the following sections schemes are developed that are more closely linked with the problem being solved and the discretisation.

4 A mesh optimisation approach

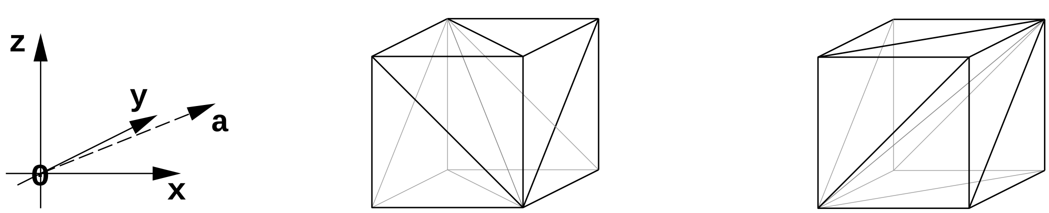
Many elliptic problems have a variational form characterised by the existence of an energy functional which must be minimised. The FE solution of these problems can also be considered as a minimisation process with respect to this functional. If the functional can be defined at the element level local mesh modifications can be performed, driven by minimisation with respect to the nodal solution and coordinates over a local patch of elements. In this way refinement/derefinement operations, edge swapping and node movement can be performed subject to the restriction that they do not increase the functional. Application of these concepts in 2D is described in [16] and [9]. Direct application of these principles to hyperbolic equations is not possible since a corresponding energy functional does not exist. However, the optimisation concept can be retained by considering alternative definitions of the objective functional to be minimised. Hence mesh quality in this context can be related to the minimisation of the functional both with respect to the nodal solution and the coordinates.

Roe [14] has described one such scheme for 2D hyperbolic problems that optimises a residual functional. This scheme can be considered as a linear least-squares FE method for the linear convection equation considered here. A global residual R is defined from the approximate solution u^h, composed of contributions from each cell k, r^h:

$$R(u,\mathbf{x}) = \frac{1}{2}\int_\Omega \left(r^h\right)^2 d\Omega = \frac{1}{2}\sum_k \int_k \left(r^h\right)^2 d\Omega, \qquad r^h = -(\mathbf{a}\cdot\nabla)\, u^h. \tag{5}$$

Minimisation of the functional R with respect to the nodal solution u_j leads to the standard least-squares FE method, however the functional can also be minimised with respect to the coordinates of the nodes in the mesh.

A simple example of the value of the functional is shown by using it to assess the quality of a mesh and its corresponding approximate solution. The total functional is evaluated over two meshes with identical nodal coordinates and solution but different connectivities, shown in Figures 4(b)-(c), with one mesh better aligned with the convection direction. The nodal solution is computed



(a) Orientation (b) Poorly aligned mesh (c) Better aligned mesh

Figure 4: Connectivities for the two meshes

with the SUPG FE scheme on a uniform 9^3 discretisation of the unit cube with connectivity shown in Figure 4(b). The misaligned mesh leads to a total functional of 85.92, whereas the more aligned mesh has a total functional of 38.94 showing clearly that the aligned mesh is preferable for this problem.

Roe considers minimising R, as defined in equation (5), with respect to the nodal coordinates $\mathbf{x}_j$, as well as the solution u_j, and uses a steepest descent approach by evaluating the functional gradients and moving in the negative direction. Here this idea is extended to the 3D problem with the following expressions for the derivatives of the functional R with respect to nodal solution u_j and coordinates $\mathbf{x}_j$:

$$\frac{\partial R}{\partial u_j} = \sum_{k\in\Omega_j} V^k \left((\mathbf{a}\cdot\nabla)\, u^h\right)_k ((\mathbf{a}\cdot\nabla)\,\phi_j)_k\,, \tag{6}$$

$$\frac{\partial R}{\partial x_j} = \sum_{k\in\Omega_j} V_k \left((\mathbf{a}\cdot\nabla)\, u^h\right)_k \left(\frac{1}{2}\left((\mathbf{a}\cdot\nabla)\, u^h\right)_k \frac{\partial \phi_j}{\partial x} - ((\mathbf{a}\cdot\nabla)\,\phi_j)_k \frac{\partial u^h}{\partial x}\right), \tag{7}$$

where V^k is the volume of cell k, and ϕ_j is the standard linear basis function at node j on cell k. The derivatives with respect to y_j and z_j follow from equation (7) by replacing x with y and z respectively. Here the coordinates are updated by a local approximate optimisation, following the approach used by Jimack and Mahmood [9] for 2D elliptic problems. The movement of the node is of course restricted due to mesh tangling considerations, hence there is a maximum distance that the node can be moved. The node is moved a fraction of this distance, e.g. 0.7, and the functional is evaluated with the new nodal position. The functional values at the original and new node positions and the gradient at the original position are used to form a local 1D quadratic problem and the nodal position is chosen by finding the functional minimum in the interval.

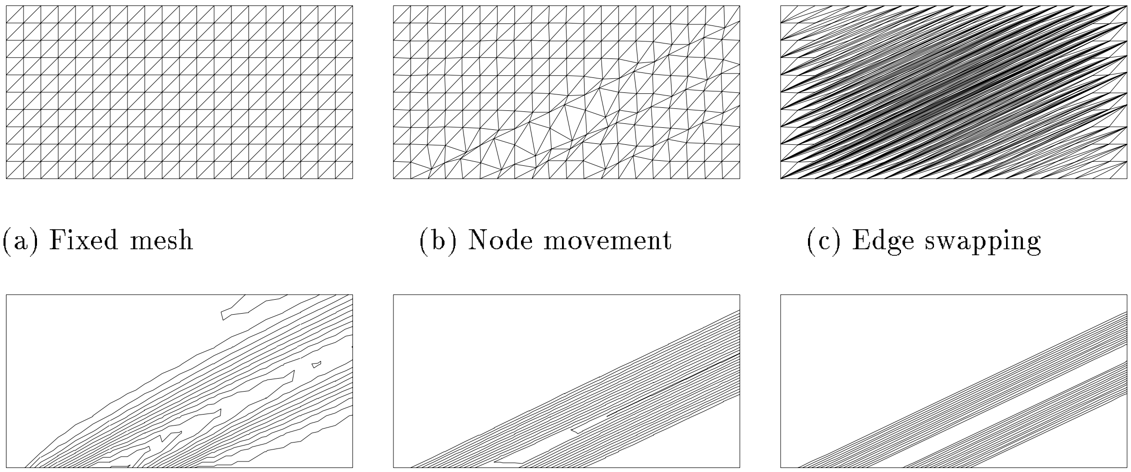
The solution update can be enhanced by solving a local problem for the nodal solution after the node has been moved. Tourigny and Hülsemann [16] consider such an approach for elliptic problems, however direct application of this technique will not work since it assumes local Dirichlet boundary conditions. For hyperbolic problems local inflow and outflow boundary conditions must be used and the solution on a local patch is used to update all the downstream nodes in the patch. Due to the hyperbolic nature of the problem the nodal updates are most effective if an upwind ordering is used, i.e. upstream nodes are considered first, since errors will be propagated downstream. Hence a Gauss-Seidel sweep through the nodes is used with the upstream nodes updated before the downstream nodes.

Roe [14] also notes the possibility that edge swapping could be used in 2D to enhance the algorithm by choosing the edge that is most aligned with the convection direction. The local definition of the functional allows local mesh operations to be considered as minimisation processes hence the decision to swap an edge can be made if this reduces the local functional values, and hence the global functional. A 2D example, shown in Figure 5, demonstrates that the combination of node movement and edge swapping leads to the exact solution in this case. Here, in 3D, the swapping operation shown in Figure 1 is performed if the sum of the element functionals for the 3 new elements is less than the sum for the 2 original elements. Table 2 shows the operations performed on the

	hr-refinement sequence					h-refinement		
	Solve	Optimise	Adapt	Solve	Optimise	Solve	Solve	Solve
Np	729		3072			729	3313	15353
R	1.016	0.381	0.381	0.189	0.130	1.016	0.428	0.180
$\|\|e\|\|_1$	0.075	0.056		0.034	0.031	0.075	0.049	0.031

Table 2: Results for the minimisation approach

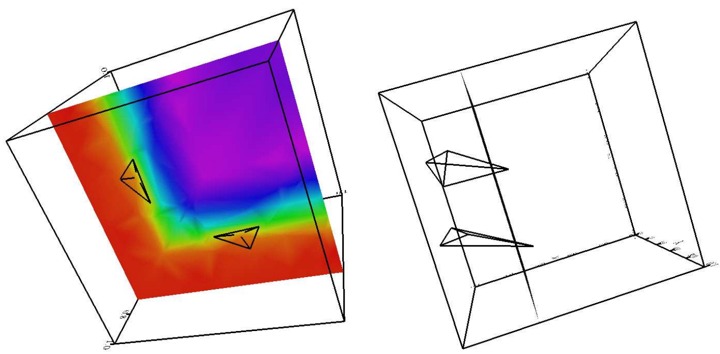
mesh, the size of the functional and the L^1 norm of the error for the model hyperbolic problem. Swapping was used however the distortion in the mesh due to the node movement prevented many swapping operations being performed and no improvement was obtained. In this case the starting



(a) Fixed mesh (b) Node movement (c) Edge swapping

(d) Solution contours (e) Solution contours (f) Solution contours

Figure 5: Comparison of 2D results for a linear convection problem



(a) Front view (b) Side view

Figure 6: Two example tetrahedra with low functional value

mesh is the one which gave the lower functional value in the previous experiment. The solution algorithm is a least-squares FE method as the SUPG method led to large functional values each time it was used, and did not combine well with this scheme. The scheme leads to low solution errors quite rapidly, but the convergence stalls after the final entry in the table. However the errors are much lower than those produced by the interpolation error scheme in Table 1 for the same resolution, and are also superior to h-refinement based on the functional value. Figure 6 shows two tetrahedra with low functional value near the layer on the first grid after optimisation. These tetrahedra have clearly been aligned with the solution by the scheme.

In 2D mesh reconnection may be used to prevent convergence stalling and it is possible that more general operations than the swapping idea considered here would allow further convergence. Note also that h-refinement does not immediately reduce the functional, since the solution is initially linearly interpolated onto the new mesh. Consequently it is not until the solution is recomputed that the correct residual is known. A directional adaptive strategy, for example based on edge bisection, could probably be used to better effect here but would still not directly minimise the functional,

5 An hr-refinement scheme based on an error estimate

The error in a cell for a linear hyperbolic equation can be decomposed into contributions from the cell and from error transported into the cell. Houston and Süli [7] have shown in 2D that for a linear hyperbolic problem the transported error can dominate. Hence residual based error indicators for hyperbolic problems may give an inaccurate representation of the error since they do not account for transport effects. For the linear hyperbolic model problem considered here a simple error estimate taking these factors into account can be derived by considering the auxiliary error equation.

Here, the error $e^h = u - u^h$ satisfies a hyperbolic problem with the residual r^h as the source term:

$$\mathbf{a} \cdot \nabla e^h \;=\; r^h \;=\; f - \mathbf{a} \cdot \nabla u^h. \tag{8}$$

Note that, due to the orthogonality of the residual and the FE test space used to solve the original flow equation, the error equation (8) must be solved either on a different mesh or by a different method to that used to obtain u^h. In 2D studies of this problem Strouboulis and Oden [15] used a Taylor-Galerkin FE method for u_h and a Discontinuous Galerkin FE method for e_h, and Houston and Süli [7] used a SUPG FE method for u_h and a Low-Diffusion Petrov-Galerkin FE method for e_h.

Since the residual r^h is defined on the cells of the mesh it is natural to use a cell-based scheme for the error equation. Here a cell-centred finite volume scheme due to Frink [5] is used. At inflow boundary faces the required flux across the face is set by computing the error using the known inflow conditions. This scheme is used to solve the error equation once a computed solution is known, and hence the residual can be evaluated.

A node movement scheme is developed by using the error in a cell, e_k, as the weight in the weighted average expression for the node position, with the coordinate increment computed as previously shown in equation (4).

$$\mathbf{x}_i^{av} \;=\; \frac{\sum_{k \in \Omega_i} |e_k| \; \mathbf{x}_k^c}{\sum_{k \in \Omega_i} |e_k|}, \qquad \mathbf{x}_i \;\rightarrow\; (1-\gamma_i)\mathbf{x}_i + \gamma_i \mathbf{x}_i^{av}, \tag{9}$$

where $\mathbf{x}_k^c$ is the centroid of cell k. Hence nodes will be attracted to regions of high error.

The performance of the error estimate can be assessed by comparing it to the h and hr-refinement schemes driven by the true error. Table 3 compares the L^1 norm of the true error for the model hyperbolic problem for these schemes. For the h-refinement scheme the error estimate leads to slightly higher errors but the results show that the errors are being uniformly reduced on each grid. The hr-refinement scheme shows similar behaviour, with higher errors than if driven by the true error, but uniformly lower errors than if the estimate is used to drive h-refinement. Comparison of the exact and estimated L^1 error norm shows that the estimate underpredicts the norm by a factor of between two and three in general. This may be expected due to numerical diffusion in the error equation, in particular on the coarse initial grid, and suggests that a more accurate solution for this equation may be required.

Note also that the finite volume scheme is constructed from fluxes through the tetrahedral faces and hence is also sensitive to mesh alignment. A face aligned parallel to the convection direction will have no flux and hence the error is not transmitted in that direction.

h-refinement				hr-refinement			
Driven by e		Driven by e^h		Driven by e		Driven by e^h	
Np	$\|\|e\|\|_1$	Np	$\|\|e\|\|_1$	Np	$\|\|e\|\|_1$	Np	$\|\|e\|\|_1$
729	0.111	729	0.111	729	0.111	729	0.111
					0.104		0.104
					0.094		0.102
2761	0.070	2799	0.079	2826	0.052	3021	0.069
					0.050		0.065
					0.046		0.062
12026	0.045	11437	0.051	12412	0.038	12260	0.043
					0.033		0.039
					0.029		0.037
52871	0.027	44105	0.032	50737	0.020	44051	0.026
					0.018		0.023
					0.016		0.022

Table 3: Comparison of the L^1 norm of the error

6 Conclusions

The results presented in the previous sections show that mesh quality and solution quality are closely linked for the hyperbolic model problem. The three approaches considered here all lead to anisotropic meshes on which the error is less than that of the isotropically refined mesh, or a similar error with fewer elements.

More complex mesh modification methods have been used by others [6] and may enhance some of these results. The mesh optimisation approach considered here would obviously benefit as any new technique could also be used to minimise the functional.

Node movement schemes are dependent on being driven by relatively accurate information. For example the error estimate is inaccurate on coarse meshes due to numerical diffusion and this may lead to problems of convergence with the overall adaptive process. However the same could also be said of the Hessian approaches which rely on recovered gradients of the computed solution. The main advantages of the Hessian approach are the explicit introduction of directional information, and that it is independent of the problem and the discretisation used and hence can be applied in many different cases. The residual estimate is dependent on the equation and on the discretisation and provides the appropriate directionality in the mesh in this case. However reformulation of the minimisation is required for different problems. The error estimate considered here is evaluated on the tetrahedral cells and contains no explicit directionality. It may be possible to extract this from the error equation by considering the edges that have large fluxes across them. Further work is required to see what, and if, more information can be extracted from these methods.

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