AN INTERIOR SURFACE GENERATION METHOD FOR ALL-HEXAHEDRAL MESHING

Tatsuhiko Suzuki[1], Shigeo Takahashi[2], Jason Shepherd[3, 4]

[1] Digital Process Ltd., 2-9-6, Nakacho, Atsugi City, Kanagawa, Japan. tsuzuki@dipro.co.jp
[2] The University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa-city, Chiba, Japan. takahashis@acm.org
[3] University of Utah, Scientific Computing and Imaging Institute, SLC, Utah, USA. jfsheph@sci.utah.edu [4]Sandia National Lab., Computational Modeling Sciences, Albuquerque, NM, USA. jfsheph@sandia.gov

ABSTRACT

This paper describes an interior surface generation method and a strategy for all-hexahedral mesh generation. It is well known that a solid homeomorphic to a ball with even number of quadrilaterals bounding the surface should be able to be partitioned into a compatible hex mesh, where each associated hex element corresponds to the intersection point of three interior surfaces. However, no practical interior surface generation method has been revealed yet for generating hexahedral meshes of quadrilateral-bounded volumes. We have deduced that a simple interior surface with at most one pair of self-intersecting points can be generated as an orientable regular homotopy, or more definitively a sweep, if the self-intersecting point types are identical, while the surface can be generated as a non-orientable one (i.e. a Möbius band) if the self-intersecting point types are distinct. A complex interior surface can be composed of simple interior surfaces generated sequentially from adjacent circuits, i.e. non-self-intersecting partial dual cycles partitioned at a self-intersecting point. We demonstrate an arrangement of interior surfaces for Schneiders' open problem, and show that for our interior surface arrangement Schneiders' pyramid can be filled with 146 hexahedral elements. We also discuss a possible strategy for practical hexahedral mesh generation.

Keywords: all-hexahedral mesh generation, interior surface arrangement, Schneiders' pyramid

1. INTRODUCTION

Product development in every industrial field is in a severe competitive phase, and FEM analysis plays an important role in product development. It is known that, in many FEM applications, hexahedral meshes give better and more effective results than tetrahedral ones. However, due to the difficulty in generating hexahedral meshes, industrial interests are shifting to hex-dominant meshes [Ow-99][Ow-01], even though all-hex meshes are still demanded because they give better results than hex-dominant ones in FEM analysis.

It is desirable to generate an all-hexahedral mesh for a model of arbitrary topological type, rather than generating a sweep-type hexahedral mesh, as is more commonly done. However, a generalized algorithm for creating such meshes does not currently exist. Techniques such as plastering [BM-93], whisker weaving [TM-95][TMB-96][FM-98], and dual cycle elimination [Mh-98] have been developed, in attempt to realize such an algorithm. Unfortunately, none of them are considered to be reliable in heavy practical use, because they can handle only a limited class of solids, and have no guarantees on the quality of the resulting meshes.

In 1995, Mitchell proposed a *hexahedral mesh existence theorem* based on an arrangement of dual-cycle-extending surfaces in the interior of solids [Th-93][Mi-95][Ep-96]. This theorem states that any simply-connected three-dimensional domain, with an even number of quadrilateral boundary faces, can be partitioned into a hexahedral mesh respecting the boundary.

However, no practical interior surface generation method has been revealed yet, and despite the proof we (and many others) have encountered several severe deadlocks in creating the topology of hexahedral mesh that conforms to a given quadrilateral boundary mesh. In this paper we present an interior surface classification theory, along with a method, that could be utilized to create actual interior surfaces, with some discussion as to how these surfaces represent a hexahedral mesh. As our later examples show, even if a topological solution exists, it may not always be an acceptable solution. Therefore, we will also discuss strategies to avoid inappropriate quadrilateral meshes when generating a hex mesh for practical analysis.

In this paper we focus on interior surfaces' properties, and leave the representation of interior surface arrangement as a future problem. The remainder of this paper is organized as follows. In Section 2, we review interior surfaces and their arrangements, and deduce the requirements for these interior surfaces for sound hexahedral mesh generation. In section 3, the classification of self-intersecting point types is discussed. In section 4 we describe a method for generating simple interior surfaces that have at most one pair of self-intersecting points. Section 5 contains a discussion of a method for composing more general interior surfaces using the simple ones. In section 6, we demonstrate a dual space solution of Schneiders' open problem based on the methods described in the previous sections. In Section 7, we conclude with a brief discussion of some future developments needed in order to generate acceptable hex meshes.

2. INTERIOR SURFACE ARRANGEMENT

2.1 Dual representation

In this paper we propose a technique using the notion of an *interior surface arrangement* [Mi-95] following the developers of the whisker weaving technique. For simplicity, we discuss hex meshing of solids homeomorphic to a ball, hereafter.

We denote a set of vertices, edges, quads and hexes as $V=\{v\}$, $E=\{e\}$, $Q=\{q\}$ and $H=\{h\}$, respectively.

For a planar graph G of a quad mesh M_Q, a *dual graph* $G^*=G^*(V^*,E^*)$ is composed as follows. A *dual vertex* v^* is placed in each quad q, and a *dual edge* e^* is placed in each edge e incident to the adjacent two dual vertices. A vertex v of a *primal graph* G is represented as a *dual face* q^* surrounded by dual edges in dual graph G^*. An edge e of a primal graph corresponds to a dual edge e^* (Fig. 1a). A sequence of dual edges connecting opposite edges in a quad (Fig. 1 b) is always closed, since the opposite edge of a quad is uniquely determined and the number of edges is finite. Thus the sequence of dual edge is called a *dual cycle* [Mh-98] (Fig. 1b). A dual cycle may self-intersect. A dual vertex is the intersection of two (local) dual cycles (Fig. 1a).

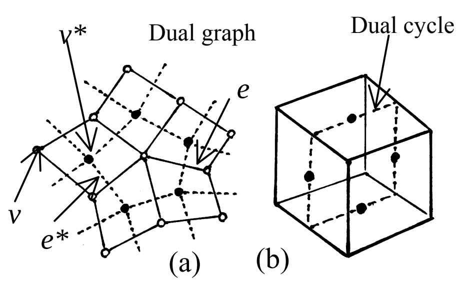


Fig. 1. (a) A Dual graph and (b) a dual cycle

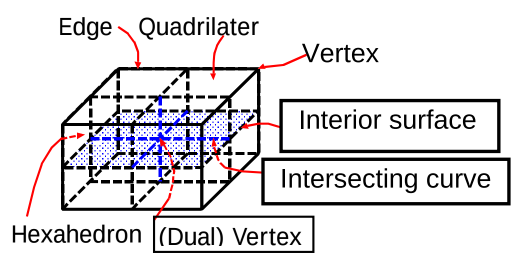


Fig. 2. Dual representation of a hex mesh

Similarly for a graph $G=G(V,E)$ of a hex mesh M_H, a dual graph $G^*=G^*(V^*,E^*)$ is constructed as follows. A dual vertex v^* is placed in each hex h, and a dual edge e^* is placed through each quad q incident to the adjacent two hexes. A hex h, quad q, edge e and vertex v of a primal graph G corresponds respectively to a dual vertex v^*, dual edge e^*, a dual face q^*, and dual polygon h^* enclosed by dual faces in dual graph G^*. A topological representation by a dual vertex *v**, dual edge *e**, *dual face q**, and dual polygon *h** is called a *dual representation* M_H *of a hex mesh M_H.

A layer of hexahedra corresponds to an *interior surface* [Ep-96]. A line, or column, of hexahedra corresponds to the intersection of two interior surfaces. A hexahedron is the dual to a vertex at the intersection of three interior surfaces in the dual representation of the hex mesh (Fig. 2).

2.2 Hex mesh existence theorem

The hexahedral mesh existence theorem is described as follows [Ep-96]:

Any simply connected three-dimensional domain with an even number of quadrilateral bounding faces can be partitioned into a hexahedral mesh respecting the boundary.

In an all-hex mesh, quads on the surface are always coupled with another quad on the surface. Therefore, a necessary condition for an all-hex mesh is that the surface be covered with even number of quads. Thus it can be shown that the number of self-intersections of dual cycles must also be even, since the intersection of distinct two dual cycles makes a pair of quads.

The proof steps of the hex mesh existent theorem are as follows [Mi-95]:

1. The surface mesh of the object is mapped onto a sphere preserving the quadrilateral mesh connectivity,
2. The quadrilateral mesh on the spherical surface forms an arrangement of dual cycles,
3. The arrangement of dual cycles is extended to an arrangement of 2D manifolds through the interior of the ball, using the theorem on "regular curve on Riemannian manifold" based on the homotopy theory [Sm-58].
4. Additional manifolds that are closed, and completely within the solid are inserted, if necessary,
5. The arrangement of 2D manifolds is dualized back to induce a hexahedral mesh.

The hex mesh existence theorem, however, addresses only the existence and does not describe how to extend dual cycles to the interior surfaces in the above-mentioned step 3. This paper attempts to propose a method to create an arrangement of interior surfaces from dual cycles, and provide a novel guideline to generate all hex meshes that attempts to use not only topological information but also geometric information. We give a solution for Schneiders' open problem as an example of utilizing an arrangement of interior surfaces.

2.3 Requirements for interior surfaces

2.3.1 Boundary

An interior surface can be bounded by one or more dual cycles, or the interior surface can also be entirely closed with no dual cycle boundary.

2.3.2 Regularity

For a collection of continuous mappings, $F(u,v)=f_v(u)$, $u \in I=[0,1]$ is called homotopy if $F(u,v)=f_v(u)$ is continuous. A parametric *regular homotopy, F,* is a homotopy where at every parameter location F is a regular curve ($F(u,v)=f_v(u)$, $u \in I$), and keeps end points and direction fixed and such that tangent vector moves continuously with the homotopy [Sm-58]. Regularity is a necessary condition to create a hex element at any point on the interior surface, and we will endeavor to create interior surfaces based on regular homotopies. If there is a *singular point* in a 3D space, a hex element layer cannot be formed, and the resulting topological structure of the hex mesh is not valid.

This can be illustrated by a simple example. If two circles C_1 and C_2, which are oriented to different directions: one is clockwise and another is counter-clockwise, on two parallel planes π_1 and π_2 are given, then a non-orientable homotopy H formed between the planes π_1 and π_2 cannot be regular (Fig. 3a), because there exists a self-intersecting curve L with the end points P_1 and P_2 on the homotopy H, and the end points P_1 and P_2 are singular. The self-intersecting curve L and an arbitrary surface ω will form a dual vertex v*, at the intersection point. The singular points P_1 and P_2 cannot be dual vertices. Thus, there exists at least one dual vertex connected to the singular points P_1 and P_2 with a self-intersecting curve. Therefore, one intersecting curve from a dual vertex is not connected with any other vertex (Fig. 3a), resulting in an invalid topology of a hex mesh.

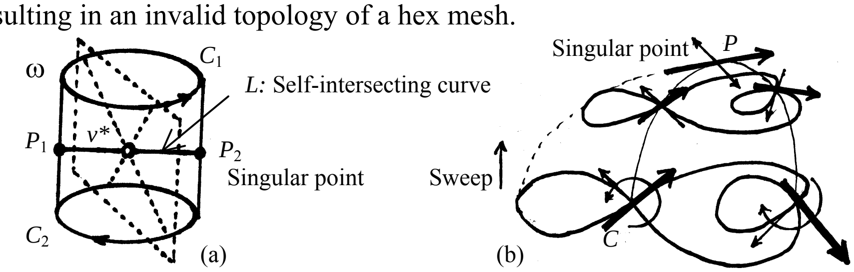


Fig. 3. A singular point makes the mesh topologically invalid.

Figure 3b depicts an orientable surface created by sweeping the curve C. The top point P is a singular point, where the tangent becomes discontinuous. Assume that the point P is a triple point and, thus, is a dual vertex. The number of the dual edges emanating from the dual vertex P is 2, which is invalid.

2.3.3 Self-intersecting point pair connectivity

Two self-intersecting points on the boundary of an interior surface should always be paired, and connected by a self-intersecting curve. This feature is called *self-intersecting point-pair connectivity*.

Regular homotopies are compatible with self-intersecting point-pair connectivity. Some self-intersecting curves of regular homotopies, however, have complicated structure. For example, the self-intersecting curve of Fig. 4 disappears from the scope at the arrow, continuing to the ‘back’. The sectional transition of the regular homotopy on the left of Fig. 4 is illustrated on the right in Fig. 4, where the self-intersecting curve is represented by a dotted line. The advancing direction changes several times. Such an interior surface will undoubtedly reduce the quality of the hex mesh, especially when the radius of curvature is small. These kinds of homotopies should be avoided.

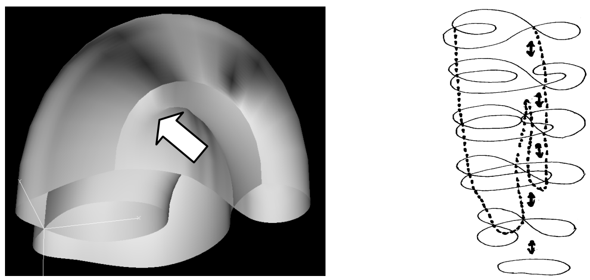


Fig. 4. A self-intersecting curve of a regular homotopy goes out of the view.

2.4 Requirement for interior surface arrangement

2.4.1 Convexity

Hex elements used in FEM must be convex, and the hexes shown in Fig. 5 are not allowed. A hex mesh is *topologically convex*, if any two quads are not incident to common edges (Fig. 5). The condition that a hex mesh M_H is convex can be described by showing the dual graph $G_F*(F*,E*)$ between the dual face set $F*$ and the dual edge set $E*$of M_H is simple. In Fig. 6 examples of convexity-lost elements in a dual space, where two dual faces f_1*,f_2* incident to two common dual edges e_1*,e_2* are illustrated.

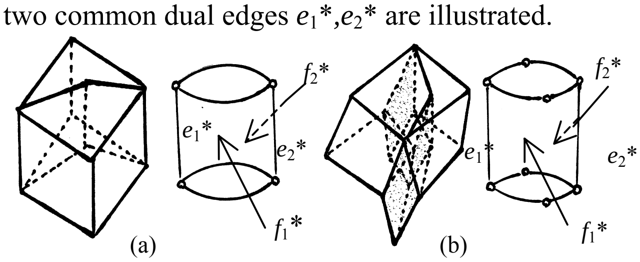


Fig. 5. Non-convex hexes

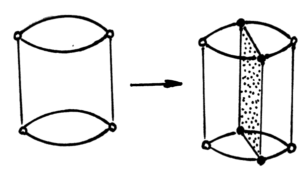


Fig. 6. Convexity recovery

Note that the simplicity of a dual graph $G_V*(V*,E*)$ between the dual vertex set $V*$ and the dual edge set $E*$ of M_H does not always ensure the simplicity of the dual graph $G_F*(F*,E*)$, where simple means that a graph has no self-loops or multi-edges. A convexity recovery operation is depicted in Fig. 6. It should also be noted that an all-hex mesh bounded by a convex quad mesh is not always topologically convex.

2.4.2 Hex element existence and connectivity

There must be at least one dual vertex on an intersecting curve between interior surfaces. Any two hex elements in a hex mesh must be connected through quads. In other words a dual graph of a hex mesh must be connected.

3. SELF-INTERSECTING POINT TYPE

Two curves with even number of self-intersecting points on a sphere are topologically deformable (Fig. 7). However, the two surfaces bounded by a curve with even number of self-intersecting points (for example in Fig. 7 the surface bounded by the first is orientable and one by the last curve is non-orientable) are not always deformable into each other. In this section the identity of self-intersecting point types are discussed in order to classify interior surface generation methods. Not only orientable surfaces but also non-orientable[1] surfaces such as a Möbius band can be interior surfaces [SZ-03], and in actuality, the identity of self-intersecting point types is closely related to orientability of interior surfaces.

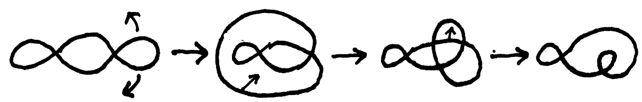


Fig. 7. Two curves with even number of self-intersecting points are deformable.

The vertices and edges of a convex polyhedron can be mapped onto a plane through a face, which is called a *window* (Fig.8), and can be transformed into a plane graph *G* and its dual graph *G**.

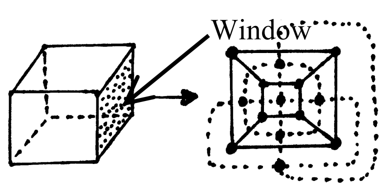


Fig. 8. Mapping to a plane graph

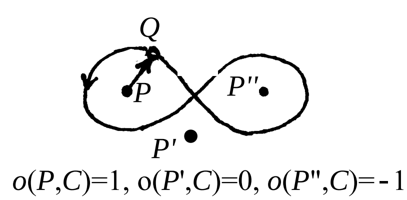


Fig. 9. Order of a point for a curve

Suppose that a plane π contains a closed curve *C* and a point *P* as shown in Fig. 9. The *order* $o(P,C)$ of the point *P* for the curve *C* is defined as the (signed) number of the vector *PQ*'s rotation around the point *P* along the curve

[1] While orientability is basically defined for only a closed surface, we can inherit and consequently define the 'orientability' of an open surface by gluing topological disks along the circles that bound the open surface to refer to the orientability of its corresponding closed surface.

C in the predefined direction by the parameter (Fig. 9). A dual cycle mapped on a plane divides the plane into several regions. There are 4 regions around a self-intersecting point partitioned by the curve, and the order of any point in a region is identical.

Fig. 10 and Fig. 11 show dual cycles with a pair of self-intersecting points and the orders of the regions around the self-intersecting points. The orders of the points in the regions around a self-intersecting point can be represented as (i, i+1, i+2, i+1), where i is called the *minimum order*. If the minimum orders of the paired self-intersecting points are equal, then the *self-intersecting point-type of the (simple) dual cycle* is said to be identical (Fig. 10), otherwise distinct (Fig. 11).

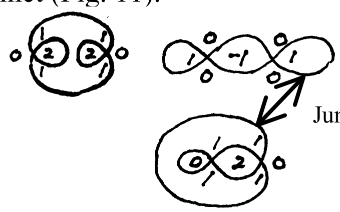


Fig. 10. Identical self-intersecting point-type

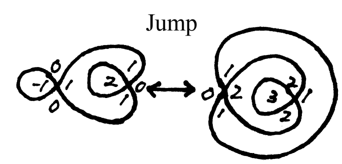


Fig. 11. Distinct self-intersecting point-type

There are three important points to note. A portion of a dual cycle can "jump" over the two self-intersecting points without crossing over it on the surface (see Fig. 10,11). This does not change the self-intersecting point-type. Secondly, a dual edge "jumps" if the window crosses over the edge composing the dual cycle. This also does not change the self-intersecting point-type. Thirdly, the self-intersecting point-type is changed if an edge crosses over one of the self-intersecting points (see Fig. 12).

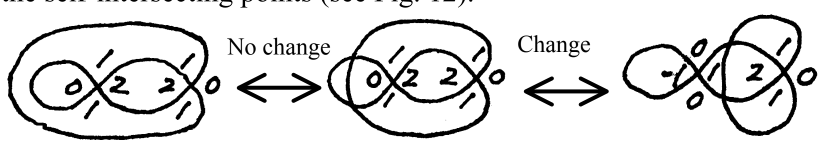


Fig. 12. Edge crossing causes a point-type change.

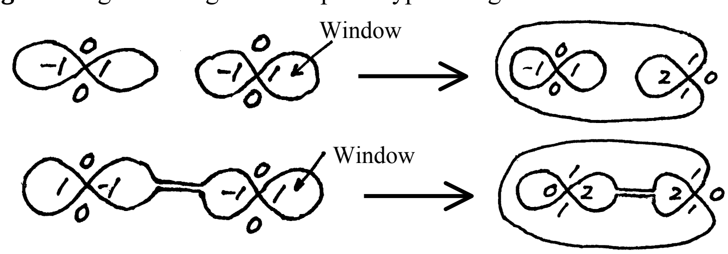


Fig. 13. Two dual cycles bounding an identical interior surface should be connected.

Two dual cycles, having a self-intersecting point respectively bounding an identical interior surface, should be connected in order to determine the self-intersecting point-type. This is due to the fact that the point-type is determined for the dual cycles respectively, and thus if the selection of the window is not appropriate, potentially erroneous results may be obtained, because an inappropriate window may "jump" only one dual cycle. A correct result will be obtained if the two dual cycles are connected into one (Fig. 13).

4. SIMPLE INTERIOR SURFACES

In this section we will describe a technique to create "simple" interior surfaces from the dual cycles containing, at most, a single pair of self-intersecting points. In the next section we will describe a technique to generate interior surfaces of two or more pairs of self-intersecting points by sequentially connecting the "simple" interior surfaces described in this section.

4.1 Simple interior surfaces bounded by a single dual cycle

In this subsection, we describe a technique to create a *simple interior surface* bounded by a single dual cycle. We define an interior surface as ‘simple’ if its bounding dual cycle(s) has (have) at most a single pair of self-intersecting points.

If the boundary of a simple interior surface is comprised of a single dual cycle, then the number of the self-intersecting points of the dual cycle has to be 0 or 2. The self-intersecting point-type is classified according to the following criteria into the following (a), (b), or (c).

(a) No self-intersection,
(b) Identical self-intersecting point type, or
(c) Different self-intersecting point type.

Based on Smale's 1958 theorem [Sm-58], Mitchell pointed out [Mi-95] that there exists a regular homotopy between a closed curve with even number of self-intersecting points and a closed curve without a self-intersection (Fig. 21).

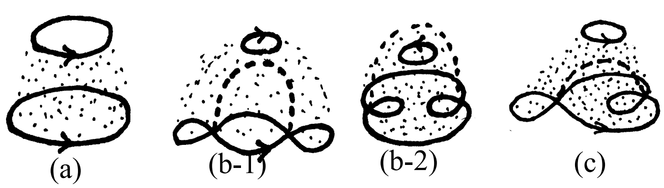


Fig. 14. Interior surface by Mitchell's proof

From an implementation standpoint, however, even if the existence of an interior surface might be proven, it is not always appropriate to create a homotopy using the method described in the proof. For example our ability to control the shape of the homotopy shown in Fig. 17 is poorer than sweep-based method. Therefore, we will consider other implementation techniques to create simple interior surfaces bounded by a single dual cycle.

However, before we consider methods for generating the interior surfaces, let us first consider some helpful constraints that must, or should, be satisfied in the creation of interior surfaces with respect to hexahedral mesh generation:

1. The interior surface bounded by a dual cycle should be nearly orthogonal to the solid surface at the boundary,
2. The interior surface should be smooth,
3. The interior surface must be completely enclosed within the solid,
4. The self-intersecting curve connecting a pair of self-intersecting points should be smooth. It is also desirable that this curve will not alternate direction multiple times (e.g. Fig. 4 is not desirable).
5. If the interior surface self-intersects, the angle of self-intersection should be nearly a right angle along the self-intersecting curve.
6. No self-intersections within the interior surface are allowed (with the exception of the possible self-intersecting curve connecting a pair of self-intersecting points).
7. The absolute value of the radius of surface curvature should be as large as possible.

Following the classification of the self-intersecting point type identity for the simple interior surface (i.e. (a), (b), or (c) from Fig. 14), we will study the implementation of generating interior surfaces to satisfy the above-mentioned requirements.

(a) Case of no self-intersecting points

First, we will define some helpful terminology. If the ruled surface created by connecting the barycenter of a dual cycle and every point of the dual cycle forms a nearly-planar surface, then the dual cycle is defined to be *planar*. Otherwise, it is defined to be *hemispherical* (Fig. 15).

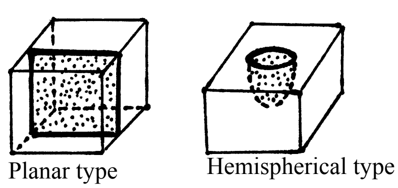


Fig. 15. Planar type and hemispherical one

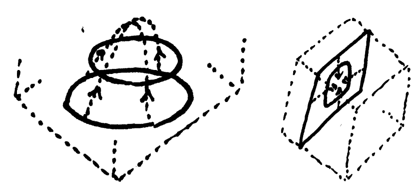


Fig. 16. Advancing closed curve

If the dual cycle is located along a closed sequence of sharp edges, a surface that is parallel to the surface of the solid is selected as the interior surface [Mh-98].

Otherwise, we can use an "*advancing closed curve method*" that repeats advancing the closed curve to the nearly normal direction of the surface, adjusting it, and finally filling the disk when the closed curve becomes small enough (Fig. 16). This method can be applied to both the planar and the hemispherical types.

(b) Case of dual cycle with identical self-intersecting point-type

For dual cycles with identical types of self-intersecting points, the homotopies used in Mitchell's proof of hex mesh existence are illustrated in Fig. 17. It is difficult to control the shapes of these homotopies, so, here we propose another form of homotopy, for which it is easier to control the shape.

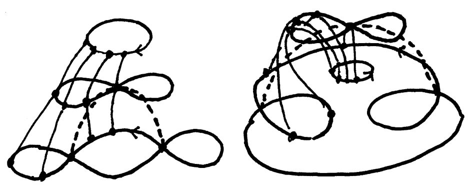


Fig. 17. Homotopy to a self-intersection free loop

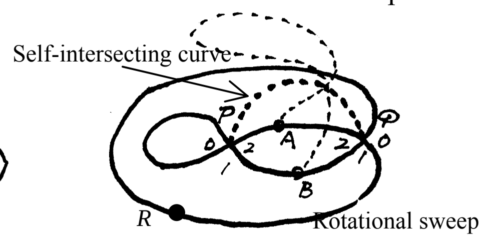


Fig. 18. Sweep type homotopy

If the points *P* and *Q,* have identical self-intersecting point types as shown in Fig. 18, then their orders are symmetric with respect to a curve passing through the points *A* and *B* on the two *paths* connecting the self-intersecting points (as shown in Fig. 18). Since the interior surface can be represented as a *sweep* of a sectional curve whose end points lie on the two paths, the corresponding swept surface is orientable. A special type of surface is a *rotational sweep* whose rotational axis passes through the points *A* and *B* (*pivots*) on the two paths.

Note that if a dual cycle is simple and the self-intersecting point-type is identical, the two points on a dual cycle cannot always be connected in an arc-wise manner without intersecting the self-intersecting curve. For example, in Fig. 18 the points *A* and *B*; *R* and *A*; *R* and *B* cannot be connected as above-mentioned manner respectively.

(c) Case of distinct self-intersecting point type

Finally, we consider the difficult case of an interior surface bounded by a curve with distinct types of self-intersecting points (Fig. 14c). Using a Möbius band instead of an orientable surface, as depicted in Fig. 19, which is homeomorphic to Fig. 14c, we can obtain an interior surface that is a regular homotopy when it is embedded in a 3D space [Fr-87]. Fig. 20 demonstrates a method to create a Möbius band, which is expressed as a regular homotopy such that the points P_0, P_1 are $f_0(0)=f_1(1)$; $f_1(0)=f_0(1)$ respectively.

For this case, it is not easy to satisfy the interior surface generation requirements by a simple method where only the end points and tangential vectors are supplied. Our experience has shown that it is quite difficult to obtain a Möbius band with such a neat transition diagram shown in Fig. 21 by a simple method automatically, as can be seen in Fig. 4, where the self-intersecting curve changes the vertical direction 3 times. Utilizing the sectional transition from the left shown in Fig. 21, however, we can reduce the number of vertical direction changes to 1.

Even though there are, indeed, topological solutions, non-orientable interior surfaces, which contains Möbius bands, should be avoided as much as possible.

Note that if a dual cycle is simple and the intersecting point types are not identical, the two points on a dual cycle cannot always be connected in an arc-wise manner without intersecting the self-intersecting curve.

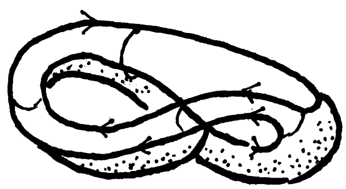


Fig. 19. Möbius band

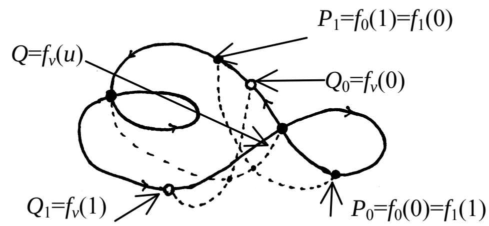


Fig. 20. Möbius band generation

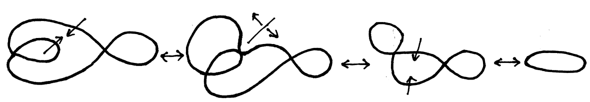


Fig. 21. Sectional transition diagram of a Möbius band to suppress the directional alternation

4.2 Simple interior surfaces bounded by two dual cycles

We will now discuss a method of generating simple interior surfaces bounded by two dual cycles, where the number of self-intersections is 0 or 2.

If an interior surface is bounded by two dual cycles, then they must also be connected by the interior surface (Fig. 22).

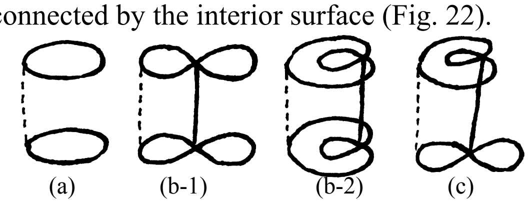


Fig. 22. Connected Dual Cycles

Consequently, the two dual cycles can be connected into one using a smooth transition. (This corresponds to the statement in Section 3 that two dual cycles with a self-intersecting point respectively bounding an interior surface should be connected prior to determining the type of the self-intersecting point.) The resulting types of self-intersecting points for generating a surface bounded by two dual cycles are the same as those in Fig. 10 and 11 of the previous section. Once the two dual cycles are converted to a closed curve, a regular homotopy can be obtained by the method described in the subsection 4.1.

5. GENERAL INTERIOR SURFACE GENERATION

We can now propose a technique to "constructively" compose a general interior surface whose dual-cycle is bounded by two or more pairs of self-intersecting points. This will be accomplished by connecting simple surfaces with at most one pair of self-intersecting points successively to form the more complex interior surfaces.

5.1 Simple interior surface decomposition

In order to create an interior surface by connecting simple interior surfaces, it is necessary to identify the pairs of self-intersecting points. In this subsection we discuss the problem of determining self-intersecting point-type combinations.

5.1.1 Circuit, triple-circuit, and basic interior surface

We define a *circuit* to be a partial dual cycle split at a self-intersecting point until it does not contain any other self-intersections. We call the splitting point a *base point* (Fig. 23). (Note that intersections between different circuits are allowed, and that the selections of circuits are not unique.) Removing specified circuits, new circuits can be specified recursively (Fig. 24).

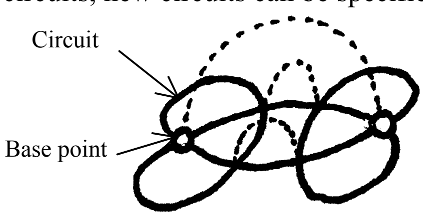


Fig. 23. Circuit and base points

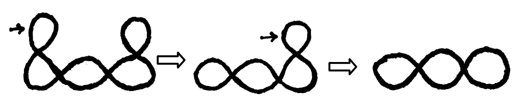


Fig. 24. Recursive specification of circuits

The simple interior surfaces described in the previous section have, at most, two self-intersecting points. Therefore, the interior surface generation method in the previous section will work for dual cycles with at most two base points. Let's call the tuple of three connected circuits a *triple-circuit* (Fig. 25). A tuple may have more then two self-intersecting points, while a simple interior surface has at most two.

We can extend a simple interior surface to a *basic interior surface* (Fig. 25) that is created by a triple-circuit, where the self-intersecting curve connecting the two base points is called the *basic self-intersecting curve* (Fig. 25). A basic interior surface may be either an “orientable” or a “non-orientable” type.

Because the triple-circuit is not always determined uniquely (see Fig. 26), the interior surface of the dual cycle shown in Fig. 26 might be orientable (left) or non-orientable (right) depending on the self-intersecting point type and the base-point selected. If the selected base points types are identical it will be orientable, otherwise it will be non-orientable.

Two circuits are considered co*incident* if they share a base point. If the circuit C_1 and C_2 are coincident, and C_2 and C_3 are coincident, then we call the two circuits C_1 and C_3 *adjacent*. For example in Fig. 25 the circuits L and R are called adjacent.

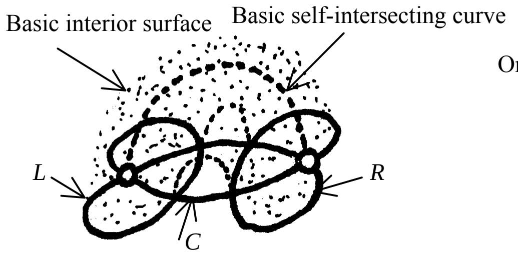


Fig. 25. Triple-circuit

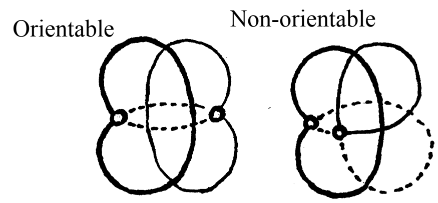


Fig. 26. Selection of a triple-circuit is not unique.

5.1.2 Interior Surface creation

We will call the operation to map a triple-circuit to a basic interior surface a "*basic interior surface creation*". Any interior surface can be created by iterating through successive basic interior surface creation operations. In other words, a basic interior surface can be created for the triple-circuit formed by cutting off existing basic interior surfaces, where we let the two circuits be coincident at the base points P and Q, by the on-surface curves connecting the two base points P and Q (Fig. 27). The on-surface curves connecting the two base points P and Q cannot intersect with existing basic self-intersecting curves (Fig. 28), because at the intersecting point four local interior surfaces meet together, which will not result in a valid hexahedral mesh.

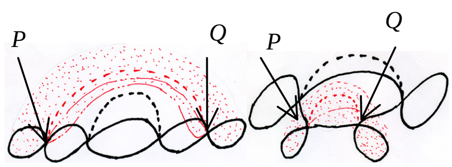


Fig. 27. Basic interior surface creation

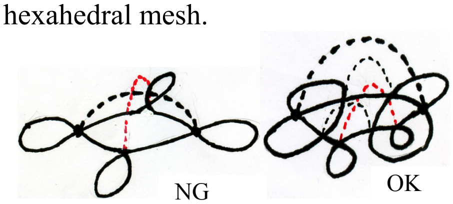


Fig. 28. Two basic self-intersecting curves cannot intersect each other.

5.2 Other self-intersecting curves

When simple interior-surface connections have been completed and the self-intersecting curves between a pair of base points have been created, it may be necessary to create other self-intersecting curves. The shape of interior surface is fixed by the creation of the other self-intersecting curves and then by dividing the faces as shown in Fig. 29.

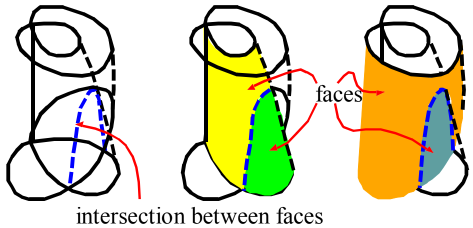
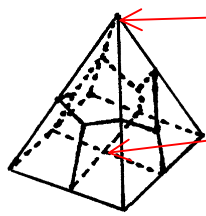


Fig. 29. Intersection between basic interior surfaces

6. SOLUTION OF SCHNEIDERS' OPEN PROBLEM

Schneiders presents [Sch-www] a problem regarding whether, or not, there exists a hexahedral mesh whose boundary exactly matches a pyramid with a prescribed surface mesh as shown in Fig. 30, and hereafter called "*Schneiders' pyramid*". (Schneiders also presents the problem of whether, or not, there exists a mesh of hexahedral elements whose boundary matches a pre-specified mesh of quadrilateral faces. Hereafter, we'll refer to this second problem as '*Schneiders' general open problem'*.) Though, several solutions have been published for Schneiders' pyramid (see [Ca-www] [YS-01]), we will attempt to solve the first problem by utilizing our "interior surface direct arrangement technique" by creating the interior surfaces directly in a dual space.



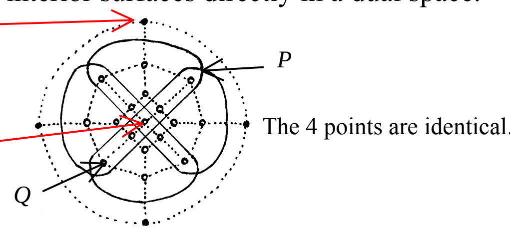


Fig. 30. Schneiders' pyramid **Fig. 31.** Triple-circuit for Schneiders' pyramid

The dual cycles of Schneiders' pyramid are shown in Fig. 31, and we let the points P and Q be selected as the base points of the tri-circuit for the dual cycle depicted with solid line. (Note that this is the same as the dual cycle depicted in Fig. 26.) The base points are selected such that the self-intersecting point-types are identical so that we can obtain an orientable surface. Then, the interior surface can be created as a rotational sweep.

Let the sweep-section rotate from the base point P to the base point Q, and let the points A and B be the pivots in Fig. 32. To obtain a regular homotopy, the sweep-vector is oriented upward at the starting point (left half), horizontally at the intermediate point (center; red), and downward at the end point (right half). Therefore, the trajectory of the point X on the sweep-section becomes a curve with a loop in blue. Fig. 33 shows the interior surface created with this curve (left whole; right section).

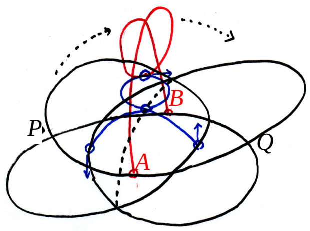


Fig. 32. Rotational sweep

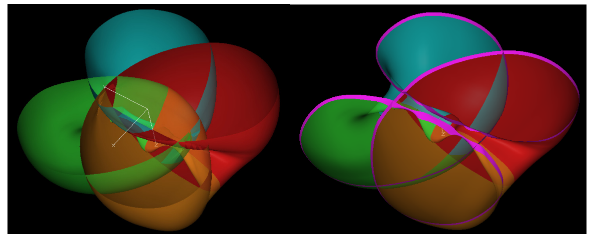


Fig. 33. Interior surface for Schneiders' pyramid

This interior surface has four features. The first is that it is regular (if the trajectory has no loop, it will make a singular point). The second is that that it is orientable. The third is that it has two triple points. (However, because this surface is orientable, it is not a Boy's surface [BE-02][Fr-87].) The fourth feature is "*through hole*" whose section is shown on the right of Fig. 33, which makes self-loops and increases the number of hexes needed to fill the volume considerably. Note that the through hole appears due to the intersection of rotational sweep section.

Fig. 33 shows a section of the interior surfaces for Schneiders' problem. Each of the triple-intersection points represents a dual vertex, and the intersecting curves are the dual edges in the dual graph of the resulting mesh. Because of the four self-loops, several of the resulting hexes require their convexity to be recovered.

In order to recover the hex convexity, we insert a cylindrical (closed) surface (depicted with dotted line as a closed interior surface in Fig. 34). The resulting intersecting curves are added to the dual graph, along with the other symmetric half of the original surface, which produces 20 dual vertices (hexes). The dual graph was represented with a prototype 3D cell-complex model, and converted to a primal graph, whose NASTRAN format data is shown in Tab.1 in the appendix together with the node numbers on the surface.

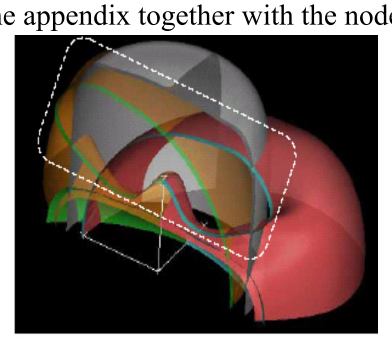


Fig. 34. Section of the interior surfaces

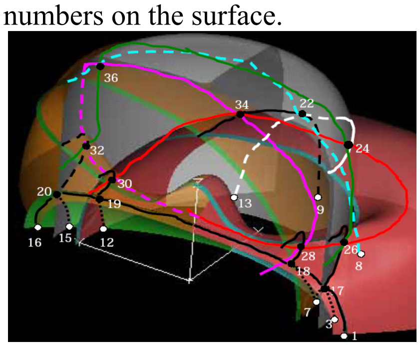


Fig. 35. Intersection between the interior surfaces for Schneiders' pyramid

Topologically, there are four dual vertices having double edges, namely 26, 28, 30, 32 (see Fig. 35). We can recover the convexity for these dual vertices by adding another cylindrical surface, resulting in an additional 18 hexes.

Furthermore, still there exist 9 convexity-taking edges, and this recovery requires 9 additional cylinders, which consequently create additional 90 hexes. For our interior surface arrangement it is confirmed that 146 dual vertices (hexes) complete the hexahedral mesh topology.

As shown above, a hex mesh can be generated from a closed quad mesh with complicated dual cycles using an “interior surface direct arrangement technique”.

Schneiders' pyramid demonstrates to us that there exist quad meshes where there is an obvious interplay between interior surfaces that result in a poorer quality mesh. The process of recovering the convexity of the elements may intrinsically require the addition of many hex elements, which inevitably reduces the industrial value of the solution.

7. CONCLUSION

7.1 Our contribution

Indeed, the hex mesh existence theory, that "Any simply connected three-dimensional domain with an even number of quadrilateral bounding faces can be partitioned into a hexahedral mesh respecting the boundary (using interior surfaces extended from dual cycles into solids)" is well known. However, a practical algorithm for generating the meshes indicated by the proof has not yet materialized. Our contribution to all-hex meshing is as follows:

We have introduced the notion of self-intersecting point-type identity for a dual cycle, and deduced that if the types of self-intersecting point pair are identical, then the interior surface will be an orientable homotopy (and can be generated via a sweep), otherwise it will be a non-orientable homotopy (i.e. a Möbius band will be contained in the interior surface).

Let a circuit be specified as partial dual cycle split by a self-intersecting point, such that it does not self-intersect. A "general" interior surface can be comprised of "simple" interior surfaces generated from adjacent circuits sequentially.

We propose to create interior surfaces by the above-mentioned method and to apply the interior surface direct arrangement technique. We claim that this is one of the solutions of Schneiders' general open problem. We have created an interior surface for Schneiders' pyramid, and showed that it can be filled by 20 hexes with convexity lost derived from the topological arrangement, or with 146 hex elements when the convexity is recovered from the topological arrangement (this does not guarantee geometric convexity).

7.2 Future Problems

7.2.1 Unstructured all-hex meshing

The following three questions with respect to unstructured all-hex mesh generation are important, where an algorithm that positively answers all three will have industrially viable solution.

1. Is the topological solution realized?
2. Can the convexity of all hexes be recovered from the topological solution in a reasonable number of hexes?
3. Are the interior surfaces of sufficient geometric quality that the hexahedral mesh also has sufficient quality?

The solution presented in this paper demonstrates an answer to the first question for the example of Schneiders' pyramid. Further research and development will be needed for a general solution.

The second question of convexity recovery for the hexes can lead to large increases in the number of hex elements, and is especially apparent with multiply self-intersecting interior surfaces. These problems have been avoided in the past by enforcing *restricted unstructured hex mesh generation* techniques that generate hex meshes in much smaller geometric and topologic domains.

The third question of how the geometric quality of the interior surface results in hex mesh quality deterioration has had little study, especially on the relationship between surface quad meshes and the interior surface generation requirements in subsection 4.1. This is also an area of future research.

7.2.2 Implementation

The surfaces generated as an example in this paper were generated without referring to the given volumetric space, and the induced topology was then mapped back into the volume. Because of the difficulty of generating interior surfaces in pre-defined volumetric boundaries (which would be required for sampling quality metrics of the resulting mesh that such interior surfaces might generate), a technique for directly arranging interior surfaces will have some difficult implementation issues. For example, surface creation with boundary constraints, geometrical surface evaluation, topological representations of the induced interior surface arrangement, etc. should be investigated. These problems are also areas for future research.

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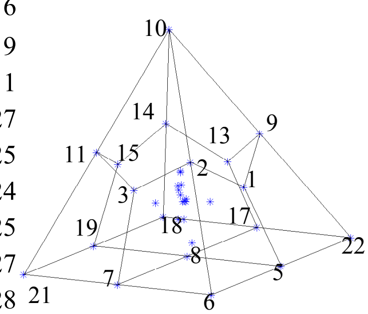
APPENDIX

Hex	N 1	N 2	N 3	N 4	N 5	N 6	N 7	N 8
17	1	2	3	4	5	6	7	8
18	9	10	11	12	1	2	3	4
19	13	14	15	16	9	10	11	12
20	17	18	19	8	13	14	15	16
21	20	3	11	15	8	7	21	19
22	22	17	8	5	9	13	23	1
23	7	8	20	3	24	25	26	27
24	5	1	23	8	24	27	26	25
25	28	4	3	27	25	8	7	24
26	27	1	4	28	24	5	8	25
27	29	12	11	30	28	4	3	27
28	30	9	12	29	27	1	4	28
29	31	16	15	32	29	12	11	30
30	32	13	16	31	30	9	12	29
31	25	8	19	33	31	16	15	32
32	33	17	8	25	32	13	16	31
33	3	20	15	11	27	26	32	30
34	1	9	13	23	27	30	32	26
35	19	15	20	8	33	32	26	25
36	17	8	23	13	33	25	26	32



Node numbers on the surface

Table 1. The 20-Hex mesh (convexity lost) by NASTRAN CHEXA Format