J Sci Comput (2012) 51:293–312
DOI 10.1007/s10915-011-9511-5

Generalised Polynomial Chaos for a Class of Linear Conservation Laws

Roland Pulch · Dongbin Xiu

Received: 13 January 2009 / Revised: 21 February 2011 / Accepted: 24 June 2011 /
Published online: 13 July 2011

Abstract Mathematical modelling of dynamical systems often yields partial differential equations (PDEs) in time and space, which represent a conservation law possibly including a source term. Uncertainties in physical parameters can be described by random variables. To resolve the stochastic model, the Galerkin technique of the generalised polynomial chaos results in a larger coupled system of PDEs. We consider a certain class of linear systems of conservation laws, which exhibit a hyperbolic structure. Accordingly, we analyse the hyperbolicity of the corresponding coupled system of linear conservation laws from the polynomial chaos. Numerical results of two illustrative examples are presented.

Keywords Generalised polynomial chaos · Galerkin method · Random parameter · Conservation laws · Hyperbolic systems

1 Introduction

In this paper, we study the impact of uncertainty on linear conservation laws, which are typically modelled as systems of hyperbolic partial differential equations (PDEs). Involved physical parameters can exhibit uncertainties. Consequently, we substitute these parameters by random variables corresponding to traditional distributions. The solution of the conservation law becomes a random process in time and space. We are interested in properties of the stochastic process like expected values and variances. Nevertheless, more sophisticated data may be required.

On the one hand, the information of the stochastic model can be obtained by a quasi Monte-Carlo simulation, for example. On the other hand, the concept of the generalised

R. Pulch (✉)
Lehrstuhl für Angewandte Mathematik und Numerische Mathematik, Bergische Universität Wuppertal, Gaußstr. 20, 42119 Wuppertal, Germany
e-mail: pulch@math.uni-wuppertal.de

D. Xiu
Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA
e-mail: dxiu@purdue.edu

polynomial chaos (gPC) yields a representation of the random process, where a separation in time-space-dependent coefficient functions and random-dependent basis polynomials is achieved. The gPC methodology, first systematically proposed in [17], is an extension of the seminal work of polynomial chaos by R. Ghanem, see [4]. It utilises orthogonal polynomials to approximate random variables in random space. The computation of the unknown expansion coefficient functions can be done either by stochastic collocation or by the solution of a larger coupled system resulting from a stochastic Galerkin method, see [16]. Accordingly, we obtain the desired information by the representation in the gPC. For an extensive review of the methodology and numerical implementations, see [14, 15].

Though the gPC Galerkin approach has been applied to a large variety of problems, its application to hyperbolic problems has been much less, largely due to the lack of theoretical understanding of the resulting system of equations. Some recent work exist [2, 5, 8, 9], most of which considered linear and scalar cases. The analysis becomes more sophisticated and involved in case of systems of hyperbolic PDEs. It becomes more challenging for nonlinear cases. An early attempt was made in [12] using gPC Galerkin technique, where the standard orthogonal polynomials as well as more sophisticated sets of basis functions are employed to facilitate the analysis.

In this article, we consider a certain class of linear systems of conservation laws. The gPC approach based on the Galerkin method yields a larger coupled system of linear PDEs, which itself represents a conservation law. We analyse the hyperbolicity of the larger system provided that the original systems are hyperbolic. Thereby, we investigate if involved matrices are real diagonalisable. A deeper understanding of this property is critical to the design of effective numerical algorithms. And here we study extensively the cases of both single random parameter and multiple random parameters. For example, an understanding of hyperbolicity of the system is important.

The article is organised as follows. We introduce linear conservation laws with random parameters in Sect. 2. The gPC approach is applied and results in a larger coupled system via the Galerkin method. In Sect. 3, we examine the hyperbolicity of the larger system of conservation laws. The case of a single random parameter as well as several random parameters is discussed. In Sect. 4, we present numerical simulations of two test examples, i.e., the wave equation and the linearised shallow water equations.

2 Problem Definition

A general nonlinear system of conservation laws in one space dimension reads

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{u}, \mathbf{p}) = \mathbf{0},$$

where the function $\mathbf{f} : \mathbb{R}^n \times \mathcal{Q} \to \mathbb{R}^n$ depends on the physical parameters $\mathbf{p} \in \mathcal{Q} \subseteq \mathbb{R}^q$. Thus the solution $\mathbf{u} : [t_0, t_1] \times [x_0, x_1] \times \mathcal{Q} \to \mathbb{R}^n$ is also parameter-dependent. The corresponding quasilinear formulation is given by

$$\frac{\partial \mathbf{u}}{\partial t} + A(\mathbf{u}, \mathbf{p})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0} \quad \text{with } A = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}.$$

Considering a solution $\mathbf{u}$ for a specific parameter tuple $\mathbf{p} \in \mathcal{Q}$, the system is called hyperbolic, if the Jacobian matrix $A(\mathbf{u}(t, x), \mathbf{p}) \in \mathbb{R}^{n\times n}$ is real diagonalisable for all involved values $\mathbf{u}(t, x)$. A hyperbolic system is called strictly hyperbolic, if the eigenvalues are always pairwise different.

We investigate a linear system of conservation laws

$$\frac{\partial \mathbf{u}}{\partial t} + A(\mathbf{p})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0} \tag{2.1}$$

with parameter-dependent matrix $A(\mathbf{p}) \in \mathbb{R}^{n\times n}$. Given a specific parameter tuple $\mathbf{p} \in \mathcal{Q}$, the system is called hyperbolic if the matrix $A(\mathbf{p})$ is real diagonalisable. Strictly hyperbolic systems are defined as above. We assume that the system (2.1) is hyperbolic for all parameters within the relevant set $\mathcal{Q}$.

Uncertainties in the parameters are modelled by independent random variables $\mathbf{p} = \boldsymbol{\xi}(\omega)$ with respect to a probability space $(\Omega, \mathcal{A}, P)$. Let each random variable exhibit a classical distribution like uniform, beta, Gaussian, etc. Thus a probability density function $\rho : \mathbb{R}^q \to \mathbb{R}$ exists, whose support is included in $\mathcal{Q}$. Given a function $f : \mathcal{Q} \to \mathbb{R}$ depending on the parameters, we denote the expected value (if exists) by

$$\langle f(\boldsymbol{\xi})\rangle := \int_\Omega f(\boldsymbol{\xi}(\omega))\,\mathrm{d}P(\omega) = \int_{\mathbb{R}^q} f(\boldsymbol{\xi})\rho(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}.$$

We employ this notation also for functions $\mathbf{f} : \mathcal{Q} \to \mathbb{R}^{m\times n}$ by components. We consider continuous random variables in this paper. For a discussion on gPC for discrete random variables, see [17].

It follows that the solution of the linear system (2.1) becomes random-dependent. We assume that this random process exhibits finite second moments, i.e., for all fixed t and x,

$$\langle u_j(t,x,\boldsymbol{\xi})^2\rangle < \infty \quad \text{for } j = 1,\dots,n. \tag{2.2}$$

Consequently, the generalised polynomial chaos (gPC), see [17], yields an expansion of the solution

$$\mathbf{u}(t,x,\boldsymbol{\xi}) = \sum_{i=0}^{\infty} \mathbf{v}_i(t,x)\Phi_i(\boldsymbol{\xi}) \tag{2.3}$$

with orthonormal basis polynomials $\Phi_i : \mathbb{R}^q \to \mathbb{R}$, i.e.,

$$\langle \Phi_i \Phi_j\rangle = \int_{\mathbb{R}^q} \Phi_i(\boldsymbol{\xi})\Phi_j(\boldsymbol{\xi})\rho(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = \delta_{ij}. \tag{2.4}$$

The family $(\Phi_i)_{i\in\mathbb{N}}$ represents a complete set of the polynomials in q variables, where the probability distribution $\rho(\boldsymbol{\xi})$ serves as the weight function in the orthogonality relation. This establishes a correspondence between the probability distribution of the input random variables $\boldsymbol{\xi}$ and the type of orthogonal polynomials. For examples, Gaussian distribution defines the Hermite polynomials, whereas uniform distribution defines the Legendre polynomials. For a detailed discussion, see [17]. The coefficient functions $\mathbf{v}_i : [t_0,t_1]\times[x_0,x_1] \to \mathbb{R}^n$ are unknown a priori. The convergence of the series (2.3) is at least pointwise in t and x due to (2.2).

We apply a finite approximation

$$\mathbf{u}^m(t,x,\boldsymbol{\xi}) := \sum_{i=0}^{m} \mathbf{v}_i(t,x)\Phi_i(\boldsymbol{\xi}). \tag{2.5}$$

Inserting (2.5) in (2.1) causes a residual $\mathbf{r}(t,x,\boldsymbol{\xi}) \in \mathbb{R}^n$. The Galerkin approach yields the condition $\langle \mathbf{r}\Phi_\ell \rangle = \mathbf{0}$ for $\ell = 0, 1, \ldots, m$. Hence we obtain the larger coupled system

$$\frac{\partial \mathbf{v}_\ell}{\partial t} + \sum_{i=0}^{m} \langle \Phi_\ell(\boldsymbol{\xi}) \Phi_i(\boldsymbol{\xi}) A(\boldsymbol{\xi}) \rangle \frac{\partial \mathbf{v}_i}{\partial x} = \mathbf{0} \quad \text{for } \ell = 0, 1, \ldots, m \tag{2.6}$$

involving the unknown coefficient functions. Using $\mathbf{v} := (\mathbf{v}_0, \ldots, \mathbf{v}_m)$, the complete system can be written as

$$\frac{\partial \mathbf{v}}{\partial t} + B \frac{\partial \mathbf{v}}{\partial x} = \mathbf{0} \tag{2.7}$$

with a matrix $B \in \mathbb{R}^{(m+1)n \times (m+1)n}$. The matrix B exhibits a block structure

$$B = \begin{pmatrix} B_{00} & \cdots & B_{0m} \\ \vdots & & \vdots \\ B_{m0} & \cdots & B_{mm} \end{pmatrix}$$

with the minors

$$B_{ij} = \langle \Phi_i(\boldsymbol{\xi}) \Phi_j(\boldsymbol{\xi}) A(\boldsymbol{\xi}) \rangle \in \mathbb{R}^{n \times n} \quad \text{for } i, j = 0, 1, \ldots, m.$$

The following analysis can be generalised directly to linear systems of conservation laws

$$\frac{\partial \mathbf{u}}{\partial t} + A(\mathbf{p}) \frac{\partial \mathbf{u}}{\partial x} = \mathbf{g}(t, x, \mathbf{u}, \mathbf{p})$$

including a source term $\mathbf{g} : [t_0, t_1] \times [x_0, x_1] \times \mathbb{R}^n \times \mathcal{Q} \to \mathbb{R}^n$, since the definition of a hyperbolic system is independent of the source term.

In case of linear PDEs, the gPC technique using the larger coupled system is significantly more efficient than a quasi Monte-Carlo simulation. For a parabolic PDE, this efficiency has been demonstrated in [10]. A challenge consists in the adequate application of the gPC for nonlinear problems, see [1]. If the coupled system (2.7) is hyperbolic, then we can use standard algorithms to solve the stochastic problem numerically. More precisely, we may apply the same methods for the system (2.7) as for the original systems (2.1). Hence numerical algorithms do not need to be adapted to the system of PDEs from the stochastic Galerkin approach.

3 Analysis of Hyperbolicity

We examine if the system (2.7) is hyperbolic, i.e., if the matrix B is real diagonalisable. Thereby, we assume that the original systems (2.1) are hyperbolic for each tuple of parameters $\mathbf{p}$ in the support of the probability density function corresponding to the random distribution.

3.1 Preliminaries

For symmetric matrices, we achieve the following theorem.

Theorem 1 *If the matrix $A(\boldsymbol{\xi})$ is symmetric for all $\boldsymbol{\xi}$ within the support of the probability density function, then the matrix B in (2.7) is real diagonalisable.*

Proof Let $B_{ij}^* \in \mathbb{R}^{n\times n}$ for $i, j = 0, 1, \ldots, m$ be the minors of the matrix $B^\top$. It follows

$$B_{ij}^* = B_{ji}^\top = \langle \Phi_j(\boldsymbol{\xi})\Phi_i(\boldsymbol{\xi})A(\boldsymbol{\xi})^\top \rangle = \langle \Phi_i(\boldsymbol{\xi})\Phi_j(\boldsymbol{\xi})A(\boldsymbol{\xi})\rangle = B_{ij}.$$

The matrix B is also symmetric. Consequently, the matrix B is real diagonalisable. □

Although mathematical modelling often yields asymmetric matrices $A(\boldsymbol{\xi})$, the corresponding hyperbolic linear conservation law can be symmetrised.

We obtain another positive result if the eigenvectors of the matrix do not depend on the parameters, i.e., the eigenvectors are not uncertain.

Theorem 2 *If the matrix $A(\boldsymbol{\xi})$ is real diagonalisable for all $\boldsymbol{\xi}$ within the support of the probability density function and the eigenvectors do not depend on $\boldsymbol{\xi}$, then the matrix B in (2.7) is real diagonalisable.*

Proof It holds $D(\boldsymbol{\xi}) = VA(\boldsymbol{\xi})V^{-1}$ with diagonal matrices $D(\boldsymbol{\xi})$ and a constant matrix V. The minors $B_{ij} \in \mathbb{R}^{n\times n}$ for $i, j = 0, 1, \ldots, m$ of the matrix B satisfy

$$\begin{aligned} B_{ij} &= \langle \Phi_i(\boldsymbol{\xi})\Phi_j(\boldsymbol{\xi})A(\boldsymbol{\xi})\rangle = \langle \Phi_i(\boldsymbol{\xi})\Phi_j(\boldsymbol{\xi})V^{-1}D(\boldsymbol{\xi})V\rangle \\ &= V^{-1}\langle \Phi_i(\boldsymbol{\xi})\Phi_j(\boldsymbol{\xi})D(\boldsymbol{\xi})\rangle V. \end{aligned}$$

Hence we write $B = (I_{m+1} \otimes V^{-1})\hat{B}(I_{m+1} \otimes V)$ using Kronecker products and the identity matrix $I_{m+1} \in \mathbb{R}^{(m+1)\times(m+1)}$. The matrix $\hat{B}$ consists of the minors $\hat{B}_{ij} = \langle \Phi_i(\boldsymbol{\xi})\Phi_j(\boldsymbol{\xi})D(\boldsymbol{\xi})\rangle$. Due to the symmetry in i, j and $D(\boldsymbol{\xi})^\top = D(\boldsymbol{\xi})$, the matrix $\hat{B}$ is symmetric. Now $\hat{B}$ and thus B is real diagonalisable. □

Theorem 1 and Theorem 2 hold for arbitrary sets of orthonormal basis functions $(\Phi_i)_{i\in\mathbb{N}}$. In case of nonlinear hyperbolic systems, the coupled system from the stochastic Galerkin technique also inherits the hyperbolicity in these two cases, which has been proven in [12].

In the following, we assume a specific structure of the matrix $A(\boldsymbol{\xi})$, which has already been considered in [11]. For $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_q)$, let

$$A(\boldsymbol{\xi}) = A_0 + \sum_{k=1}^{q} \eta_k(\xi_k)A_k \tag{3.1}$$

with constant matrices $A_0, A_1, \ldots, A_k \in \mathbb{R}^{n\times n}$ and nonlinear scalar functions $\eta_k : \mathbb{R} \to \mathbb{R}$. This structure allows for a specific analysis in contrast to the general form. However, linear hyperbolic systems often exhibit matrices of the form (3.1) with respect to the involved parameters in the applications.

Without loss of generality, we assume $\langle \eta_k(\xi_k)\rangle = 0$ and $\langle \eta_k(\xi_k)^2\rangle = 1$ for each $k = 1, \ldots, q$. (Note this can always be achieved by properly shifting and scaling the matrices.) Observing (3.1), the matrix A_0 is seen as a constant part, whereas the sum represents a perturbation. The magnitude of the perturbation is specified by the norm of the matrices $A_1, \ldots, A_k$.

Using (3.1), it follows

$$B_{ij} = \langle \Phi_i(\boldsymbol{\xi})\Phi_j(\boldsymbol{\xi})A(\boldsymbol{\xi})\rangle = \delta_{ij}A_0 + \sum_{k=1}^{q} \langle \eta_k(\xi_k)\Phi_i(\boldsymbol{\xi})\Phi_j(\boldsymbol{\xi})\rangle A_k.$$

Now the coupled system (2.6) reads

$$\frac{\partial \mathbf{v}_\ell}{\partial t} + \sum_{i=0}^{m} \left[\delta_{il} A_0 + \sum_{k=1}^{q} \langle \eta_k(\xi_k) \Phi_i(\boldsymbol{\xi}) \Phi_\ell(\boldsymbol{\xi}) \rangle A_k \right] \frac{\partial \mathbf{v}_i}{\partial x} = \mathbf{0}$$

or, equivalently,

$$\frac{\partial \mathbf{v}_\ell}{\partial t} + A_0 \frac{\partial \mathbf{v}_\ell}{\partial x} + \sum_{k=1}^{q} A_k \left[\sum_{i=0}^{m} \langle \eta_k(\xi_k) \Phi_i(\boldsymbol{\xi}) \Phi_\ell(\boldsymbol{\xi}) \rangle \frac{\partial \mathbf{v}_i}{\partial x} \right] = \mathbf{0} \tag{3.2}$$

for $\ell = 0, 1, \ldots, m$. As an abbreviation, we define the matrices

$$S_k = (\sigma_{ij}^k) \in \mathbb{R}^{(m+1)\times(m+1)}, \quad \sigma_{ij}^k := \langle \eta_k(\xi_k) \Phi_i(\boldsymbol{\xi}) \Phi_j(\boldsymbol{\xi}) \rangle.$$

The coupled system (2.7) can be written in the form

$$\frac{\partial \mathbf{v}}{\partial t} + \left[I_{m+1} \otimes A_0 + \sum_{k=1}^{q} S_k \otimes A_k \right] \frac{\partial \mathbf{v}}{\partial x} = \mathbf{0} \tag{3.3}$$

with Kronecker products and the identity matrix $I_{m+1} \in \mathbb{R}^{(m+1)\times(m+1)}$.

Each matrix S_k is symmetric and thus real diagonalisable, i.e.,

$$S_k = T_k D_k T_k^\top$$

with orthogonal matrices T_k and diagonal matrices D_k. Remark that the transformation matrices T_k are not identical for different k in general.

3.2 Single Random Parameter

We examine the special case of just one random parameter. In the original system (2.1), the dependence on the parameter reads

$$A(p_1) = A_0 + \eta_1(p_1) A_1.$$

Let ρ be the density function of the probability distribution assigned to p_1. We assume that $A(p_1)$ is real diagonalisable for all $p_1 \in \operatorname{supp}(\rho)$ (support of the density function), i.e., the system (2.1) is hyperbolic for each parameter $p_1 \in \operatorname{supp}(\rho)$.

The coupled system (3.3) becomes

$$\frac{\partial \mathbf{v}}{\partial t} + \left[I_{m+1} \otimes A_0 + S_1 \otimes A_1 \right] \frac{\partial \mathbf{v}}{\partial x} = \mathbf{0}. \tag{3.4}$$

In this case, we achieve a positive result concerning the hyperbolicity applying an arbitrary set $(\Phi_i)_{i\in\mathbb{N}}$ of basis functions.

Theorem 3 *Let $A(p_1)$ be real diagonalisable for all $p_1 \in \operatorname{supp}(\rho)$. If the eigenvalues λ_ℓ of the matrix S_1 satisfy $\lambda_\ell \in \mathcal{G}$ for all ℓ with*

$$\mathcal{G} := \{\eta_1(p_1) : p_1 \in \operatorname{supp}(\rho)\},$$

then the system (3.4) is hyperbolic.

Proof Applying the multiplication rule $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$, we transform the system (3.4) into

$$(T_1 \otimes I_n)\frac{\partial \mathbf{v}}{\partial t} + \left[(T_1 T_1^\top) \otimes A_0 + (T_1 S_1 T_1^\top) \otimes A_1\right](T_1 \otimes I_n)\frac{\partial \mathbf{v}}{\partial x} = \mathbf{0}.$$

The substitution $\mathbf{w} := (T_1 \otimes I_n)\mathbf{v}$ yields the equivalent system

$$\frac{\partial \mathbf{w}}{\partial t} + \left[I_{m+1} \otimes A_0 + D_1 \otimes A_1\right]\frac{\partial \mathbf{w}}{\partial x} = \mathbf{0}.$$

Let $C := I_{m+1} \otimes A_0 + D_1 \otimes A_1$ and $D_1 = \operatorname{diag}(\lambda_0, \ldots, \lambda_m)$. Consequently, the matrix C exhibits a block diagonal structure with the minors

$$C_\ell = A_0 + \lambda_\ell A_1 \quad \text{for } \ell = 0, 1, \ldots, m. \tag{3.5}$$

Since we assume that $\lambda_\ell \in \mathcal{G}$ holds, the matrix C_ℓ is real diagonalisable for each ℓ. It follows that the total matrix C is real diagonalisable. □

In particular, the assumption $\lambda_\ell \in \mathcal{G}$ used in Theorem 3 is always satisfied in case of a Gaussian distribution ($\operatorname{supp}(\rho) = \mathbb{R}$) provided that η_1 is surjective. The function η_1 is surjective in the important case $\eta_1(p_1) \equiv p_1$, for example.

We consider the special case $\eta_1(p_1) \equiv p_1$, where the corresponding matrix is given by $S_1 = (\langle \xi \Phi_i \Phi_j \rangle)$. Thus the following result applies to the particular case $A(\xi) = A_0 + \xi A_1$.

Lemma 1 *Let ξ be a random variable with the probability density function $\rho(\xi)$. Let $\{\Phi_i(\xi)\}_{i=0}^m$ be the gPC orthogonal polynomials satisfying*

$$\langle \Phi_i \Phi_j \rangle = \int_{\mathbb{R}} \Phi_i(\xi)\Phi_j(\xi)\rho(\xi)\, d\xi = \delta_{ij}. \tag{3.6}$$

Then the eigenvalues of the symmetric matrix $S_1 \in \mathbb{R}^{(m+1)\times(m+1)}$ are the zeros of the $m+1$ degree polynomial $\Phi_{m+1}(\xi)$.

Proof It is well known that the univariate orthonormal polynomials from (3.6) satisfy a three-term recurrence relation in the following form

$$\xi \Phi_k(\xi) = b_{k+1}\Phi_{k+1}(\xi) + a_k \Phi_k(\xi) + b_k \Phi_{k-1}(\xi), \quad k = 0, 1, 2, \ldots$$

with $\Phi_{-1}(\xi) = 0$ and $\Phi_0(\xi) = 1$ and a_k, b_k are real numbers satisfying certain conditions. Let us consider the polynomials of degree up to m. By using matrix-vector notation, we denote $\mathbf{\Phi}(\xi) = (\Phi_0(\xi), \ldots, \Phi_m(\xi))^\top$ and rewrite the three-recurrence relation for up to m as

$$\xi \mathbf{\Phi}(\xi) = J\mathbf{\Phi}(\xi) + b_{m+1}\Phi_{m+1}(\xi)\mathbf{e}_{m+1}.$$

The tridiagonal symmetric matrix $J \in \mathbb{R}^{(m+1)\times(m+1)}$ takes the form

$$J = \begin{pmatrix} a_0 & b_1 & & & \\ b_1 & a_1 & b_2 & & \\ & b_2 & a_2 & \ddots & \\ & & \ddots & \ddots & b_m \\ & & & b_m & a_m \end{pmatrix}$$

and $\mathbf{e}_{m+1} = (0, \dots, 0, 1)^\top$ is the unit vector of length $m+1$. It is now obvious that if ξ_i for $i = 1, \dots, m+1$ are the zeros of the polynomial $\Phi_{m+1}(\xi)$, then the above matrix equation becomes an eigenvalue problem for J. Therefore, the eigenvalues of matrix J are the zeros of the polynomial $\Phi_{m+1}(\xi)$. On the other hand, by using the orthonormality (3.6) and the three-term recurrence, it follows

$$\langle \xi \Phi_i \Phi_j \rangle = b_{i+1} \langle \Phi_{i+1} \Phi_j \rangle + a_i \langle \Phi_i \Phi_j \rangle + b_i \langle \Phi_{i-1} \Phi_j \rangle$$

and thus $S_1 = J$. This completes the proof. □

We remark that though similar results were presented in [13] for several well-known orthogonal polynomials, the above proof is rooted on the work of [3], which is more general and elegant.

3.3 Multiple Random Parameters

Now we investigate the general case of $q \geq 2$ random parameters. The corresponding system is given in (3.2). In the gPC, the multivariate basis polynomials read

$$\Phi_{i_1,\dots,i_q}(\xi_1, \dots, \xi_q) := \prod_{\ell=1}^{q} \Psi_{i_\ell}^\ell(\xi_\ell),$$

where Ψ_i^ℓ represents the univariate basis polynomial of degree i corresponding to the ℓth random parameter. As an abbreviation, we apply $\mathbf{i} := (i_1, \dots, i_q)$. Let $\langle \cdot \rangle_\ell$ denote the expected value of a random variable depending on the parameter ξ_ℓ only. It holds due to the independence of the random parameters

$$\begin{aligned}\langle \Phi_{\mathbf{i}} \Phi_{\mathbf{j}} \rangle &= \left\langle \left(\prod_{\ell=1}^{q} \Psi_{i_\ell}^\ell(\xi_\ell) \right) \left(\prod_{k=1}^{q} \Psi_{j_k}^k(\xi_k) \right) \right\rangle = \left\langle \prod_{\ell=1}^{q} \left(\Psi_{i_\ell}^\ell(\xi_\ell) \Psi_{j_\ell}^\ell(\xi_\ell) \right) \right\rangle \\ &= \prod_{\ell=1}^{q} \left\langle \Psi_{i_\ell}^\ell(\xi_\ell) \Psi_{j_\ell}^\ell(\xi_\ell) \right\rangle_\ell = \prod_{\ell=1}^{q} \delta_{i_\ell j_\ell} =: \delta_{\mathbf{ij}},\end{aligned}$$

which confirms the orthogonality of the basis functions, cf. (2.4).

We consider two different sets of basis polynomials

$$\mathcal{M}_R := \left\{ \Phi_{\mathbf{i}} : \sum_{\ell=1}^{q} i_\ell \leq R \right\} \quad \text{and} \quad \mathcal{N}_R := \left\{ \Phi_{\mathbf{i}} : \max_{1 \leq \ell \leq q} i_\ell \leq R \right\}$$

for each degree $R \in \mathbb{N}$. The set $\mathcal{M}_R$ represents all multivariate polynomials up to degree R as used in a Taylor expansion. We will provide a counterexample with two random parameters in Sect. 4.2, which demonstrates that the corresponding coupled system (3.2) is not always hyperbolic using $\mathcal{M}_R$, although the underling systems (2.1) are all hyperbolic. The counterexample exhibits the specific form (3.1) with two linear functions η_1, η_2. Nevertheless, it also represents a counterexample for the general case (2.1).

For the set $\mathcal{N}_R$, we define

$$|\mathbf{i}| := \max_{1 \leq \ell \leq q} i_\ell$$

and the system (3.2) reads

$$\frac{\partial \mathbf{v_l}}{\partial t} + A_0 \frac{\partial \mathbf{v_l}}{\partial x} + \sum_{k=1}^{q} A_k \left[\sum_{|\mathbf{i}|=0}^{R} \langle \eta_k(\xi_k) \Phi_{\mathbf{i}}(\boldsymbol{\xi}) \Phi_{\mathbf{l}}(\boldsymbol{\xi}) \rangle \frac{\partial \mathbf{v_i}}{\partial x} \right] = \mathbf{0}$$

for $|\mathbf{l}| \leq R$. The involved expected value can be calculated as

$$\begin{aligned} \langle \eta_k(\xi_k) \Phi_{\mathbf{i}}(\boldsymbol{\xi}) \Phi_{\mathbf{l}}(\boldsymbol{\xi}) \rangle &= \left\langle \eta_k(\xi_k) \left(\prod_{\alpha=1}^{q} \Psi_{i_\alpha}^{\alpha}(\xi_\alpha) \right) \left(\prod_{\beta=1}^{q} \Psi_{\ell_\beta}^{\beta}(\xi_\beta) \right) \right\rangle \\ &= \left\langle \eta_k(\xi_k) \Psi_{i_k}^{k}(\xi_k) \Psi_{\ell_k}^{k}(\xi_k) \right\rangle_k \prod_{\alpha \neq k} \left\langle \Psi_{i_\alpha}^{\alpha}(\xi_\alpha) \Psi_{\ell_\alpha}^{\alpha}(\xi_\alpha) \right\rangle_\alpha \\ &= \left\langle \eta_k(\xi_k) \Psi_{i_k}^{k}(\xi_k) \Psi_{\ell_k}^{k}(\xi_k) \right\rangle_k \prod_{\alpha \neq k} \delta_{i_\alpha \ell_\alpha}. \end{aligned}$$

We define the matrices $S_k := (\langle \eta_k(\xi_k) \Psi_{i_k}^{k}(\xi_k) \Psi_{\ell_k}^{k}(\xi_k) \rangle)$ for $0 \leq i_k, \ell_k \leq R$ again. Hence $S_k \in \mathbb{R}^{(R+1)\times(R+1)}$ holds for all k. Using an adequate ordering of the basis polynomials and $\hat{\mathbf{v}} = (\mathbf{v_l})_{|\mathbf{l}| \leq R}$, the above system reads

$$\frac{\partial \hat{\mathbf{v}}}{\partial t} + \left(I_{(R+1)^q} \otimes A_0 + \sum_{k=1}^{q} N_k \otimes A_k \right) \frac{\partial \hat{\mathbf{v}}}{\partial x} = \mathbf{0} \tag{3.7}$$

with the matrices

$$\begin{aligned} N_k &:= I_{R+1} \otimes \cdots \otimes I_{R+1} \otimes S_k \otimes I_{R+1} \otimes \cdots \otimes I_{R+1} \\ &= I_{(R+1)^{(k-1)}} \otimes S_k \otimes I_{(R+1)^{(q-k)}}. \end{aligned}$$

To analyse the hyperbolicity, we consider the matrices

$$A(\mu_1, \ldots, \mu_q) = A_0 + \sum_{k=1}^{q} \mu_k A_k \tag{3.8}$$

in the original systems (2.1). Let $\mathcal{G}$ be a q-dimensional cuboid of the form

$$\mathcal{G} = \prod_{k=1}^{q} G_k \tag{3.9}$$

with $G_k = [a_k, b_k]$, $G_k = [a_k, +\infty)$, $G_k = (-\infty, b_k]$ or $G_k = \mathbb{R}$. We assume that the matrices (3.8) are real diagonalisable for all $\boldsymbol{\mu} \in \mathcal{G}$.

Theorem 4 *Let $\lambda_{k,\ell}$ be the eigenvalues of the matrix S_k. If $\lambda_{k,\ell} \in G_k$ holds for each ℓ, then the coupled system (3.2) based on the basis functions $\mathcal{N}_R$ is hyperbolic.*

Proof Each matrix S_k is symmetric and thus diagonalisable, i.e., $S_k = T_k D_k T_k^{\top}$. In the following, we apply the multiplication rule

$$(A_1 \otimes A_2 \otimes \cdots \otimes A_r)(B_1 \otimes B_2 \otimes \cdots \otimes B_r) = (A_1 B_1) \otimes (A_2 B_2) \otimes \cdots \otimes (A_r B_r).$$

We arrange the transformation matrix

$$\hat{T} := T_1 \otimes T_2 \otimes \cdots \otimes T_q.$$

It holds

$$\hat{T}^{-1} = T_1^\top \otimes T_2^\top \otimes \cdots \otimes T_q^\top = \hat{T}^\top.$$

We perform a similarity transformation of the matrix in the system (3.7)

$$\begin{aligned} C &:= (\hat{T}^\top \otimes I_n)\left(I_{(R+1)^q} \otimes A_0 + \sum_{k=1}^{q} N_k \otimes A_k\right)(\hat{T} \otimes I_n) \\ &= (\hat{T}^\top \hat{T}) \otimes A_0 + \sum_{k=1}^{q} (\hat{T}^\top N_k \hat{T}) \otimes A_k. \end{aligned}$$

It follows

$$\begin{aligned} \hat{T}^\top N_k \hat{T} &= T_1^\top T_1 \otimes \cdots \otimes T_{k-1}^\top T_{k-1} \otimes T_k^\top S_k T_k \otimes T_{k+1}^\top T_{k+1} \otimes \cdots \otimes T_q^\top T_q \\ &= I_{(R+1)^{(k-1)}} \otimes D_k \otimes I_{(R+1)^{(q-k)}} =: \hat{D}_k, \end{aligned}$$

where $\hat{D}_k$ is a diagonal matrix of order $(R+1)^q$ containing only diagonal elements from D_k. Thus the transformed matrix becomes

$$C = I_{(R+1)^q} \otimes A_0 + \sum_{k=1}^{q} \hat{D}_k \otimes A_k.$$

This matrix is block diagonal with the minors

$$C_\ell := A_0 + \sum_{k=1}^{q} \lambda_{k,\ell} A_k$$

for $\ell = 1, \ldots, (R+1)^q$. Each coefficient $\lambda_{k,\ell}$ is an eigenvalue of the symmetric matrix S_k. It follows that each matrix C_ℓ is real diagonalisable due to the assumption $\lambda_{k,\ell} \in G_k$. □

Again we can guarantee the assumption made by Theorem 4 in the special case $\eta_k(p_k) \equiv p_k$ for all k provided that the original systems (2.1) are hyperbolic for all $\mathbf{p} \in \operatorname{supp}(\rho)$. Using a Gaussian distribution for p_k yields $G_k = \mathbb{R}$. Given a uniform distribution, it follows $G_k = [a_k, b_k]$.

Theorem 4 implies the hyperbolicity of the gPC system (3.7), where a set $\mathcal{N}_R$ of basis functions is involved. The counterexample given in Sect. 4.2 yields that the coupled system (2.6) is not always hyperbolic in case of the set $\mathcal{M}_R$. Nevertheless, in view of $\mathcal{M}_R \subset \mathcal{N}_R$, we can always enlarge the set of basis polynomials to guarantee a hyperbolic system. Due to $|\mathcal{N}_R| = (R+1)^q$ and $|\mathcal{M}_R| = \frac{(R+q)!}{R!q!}$, we obtain $|\mathcal{N}_R| \approx q!|\mathcal{M}_R|$. Hence the sizes $|\mathcal{N}_R|n$ and $|\mathcal{M}_R|n$ of the corresponding systems of conservation laws differ significantly for large numbers of random parameter.

Finally, we comment shortly on the case of small random perturbations, which often yields stronger results, cf. [11]. The standardisation $\langle \eta_j(\xi_j)^2 \rangle = 1$ for $j = 1, \ldots, q$ implies

that the magnitude of the stochastic perturbation is included in the matrices $A_1, \ldots, A_q$. If it holds

$$\|A_j\| \to 0 \quad \text{for all } j = 1, \ldots, q,$$

then the matrix from (3.3) results to $I_{m+1} \otimes A_0$ in the limit case. It follows that the system (3.3) is not strictly hyperbolic in the limit, since multiple eigenvalues occur. For a small random perturbation, we obtain a matrix $I_{m+1} \otimes A_0 + E$ with $\|E\| \ll 1$. Gerschgorin's theorem implies that the eigenvalues of this matrix are located within small circles around the real eigenvalues of A_0 in the complex plane. However, since multiple eigenvalues appear in the limit, pairs of conjugate complex eigenvalues are not excluded for the perturbed matrix. Hence we do not achieve more information on hyperbolicity in the case of small random perturbations.

4 Numerical Simulation

We discuss two test examples, which exhibit a single random parameter and two random parameters, respectively.

4.1 Wave Equation

As illustrative example, we consider the scalar wave equation in one space dimension

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \quad \text{with } w : [0, T] \times \mathbb{R} \to \mathbb{R},\ (t, x) \mapsto w(t, x) \tag{4.1}$$

and velocity $c > 0$. The corresponding initial values read

$$w(0, x) = w_0(x), \qquad \left.\frac{\partial w}{\partial t}\right|_{t=0} = w_1(x),$$

where w_0, w_1 are predetermined functions. Using $u_1 := \frac{\partial w}{\partial x}$ and $u_2 := \frac{\partial w}{\partial t}$, the equivalent system of first order is given by

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4.2}$$

see [6]. The according initial values result to

$$u_1(0, x) = w_0'(x), \qquad u_2(0, x) = w_1(x).$$

For $\mathbf{u} := (u_1, u_2)^\top$, the system (4.2) exhibits the form (2.1) with a matrix $A(c)$ depending on the velocity c.

We apply the initial conditions

$$w_0(x) = \begin{cases} (x-1)^2(x+1)^2 & \text{for } -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

and $w_1 \equiv 0$. Hence both w_0 and w_0' are smooth functions. We solve the system (4.2) with $c = 1$ using the Lax-Wendroff scheme, see [7]. A grid in the domain $x \in [-5, 5]$ and

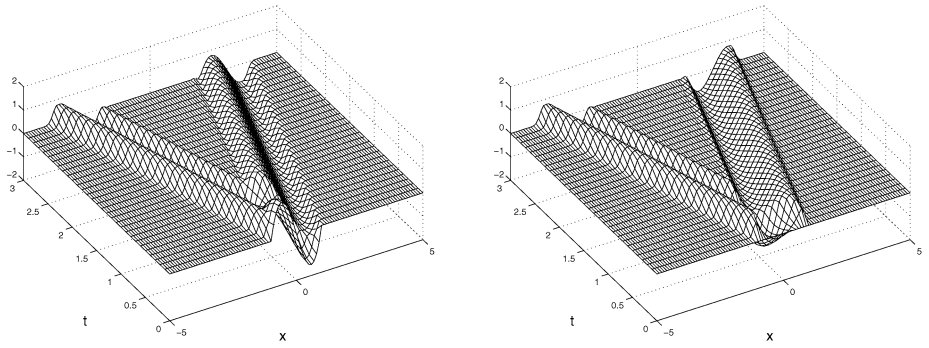
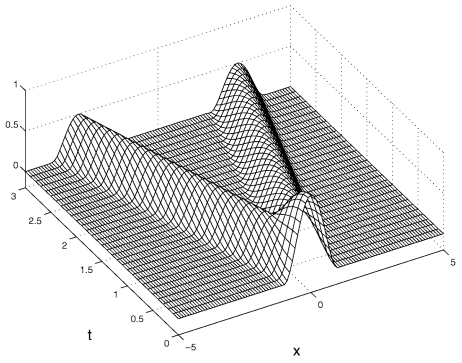


Fig. 1 Solutions u_1 (*left*) and u_2 (*right*) of system (4.2) for $c = 1$

Fig. 2 Solution of wave equation (4.1) for $c = 1$



$t \in [0, 3]$ is arranged with the mesh sizes $\Delta x = \frac{1}{40}$ and $\Delta t = \frac{1}{80}$. Thus the CFL condition is satisfied, which is necessary for the stability of the method, see [7]. We apply the boundary conditions $\mathbf{u}(-5, t) = \mathbf{u}(5, t) = \mathbf{0}$. Figure 1 illustrates the resulting solutions. We compute the corresponding solution of (4.1) by integration of the partial derivatives obtained from (4.2), see Fig. 2. We remark that there exist many other choices of spatial and temporal discretisations. The key is to ensure grid resolution independent results. Here our focus is on the random domain and the model problems usually have discontinuity in random domain but not in physical domain. The choice of the Lax-Wendroff scheme was tested and shown to be sufficient. For problems with more complex nature, more sophisticated schemes can be used.

Now we arrange a random velocity via

$$c(\xi) = 1 + \alpha\xi$$

with a constant $\alpha \in \mathbb{R}$. A uniformly distributed random variable $\xi \in [-1, 1]$ is used. Consequently, the matrix $A(c)$ depends continuously on a random parameter. We apply a random distribution for the velocity c and not for c^2 to achieve a nonlinear dependence in $A(c)$, i.e., to investigate the more complex case. In the following, we choose $\alpha = 0.1$, which corresponds to variations of 10% in the velocity.

Due to the uniform distribution, the gPC applies the Legendre basis, see [17]. Since no discontinuities appear in random space, we expect an exponential convergence of the gPC expansion (2.3). The eigenvalues of the matrices in the larger coupled system (2.6) are shown

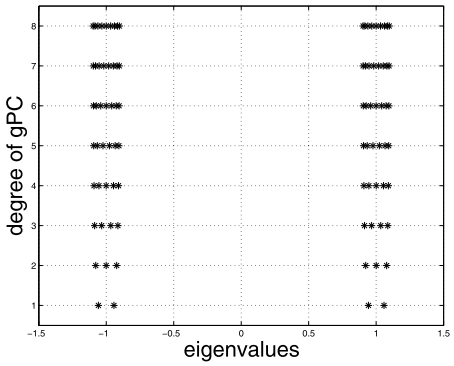


Fig. 3 Eigenvalues of matrix in gPC systems for wave equation

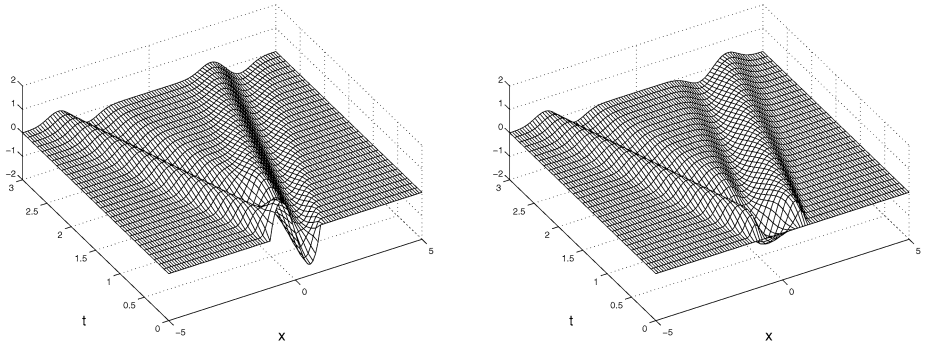


Fig. 4 Expected values for u_1 (*left*) and u_2 (*right*) resulting from gPC for wave equation

in Fig. 3 for different degrees m. In the following, we employ the univariate basis polynomials up to the degree $m = 4$. We solve the corresponding initial-boundary value problem in the same domain with the same mesh sizes as above. Again the Lax-Wendroff scheme yields the numerical solution. Figure 4 demonstrates the approximations of the expected values achieved by the gPC, i.e., the first coefficient functions. These expected values are similar to the deterministic solution in case of $c = 1$. The corresponding approximations of the variance are shown in Fig. 5. For a more detailed visualisation, Fig. 6 depicts the expected values and the variances at the final time. Furthermore, Fig. 7 illustrates the other coefficient functions of the first component of the solution. The behaviour of the coefficient functions of the second component is similar.

Next, we observe the convergence of the gPC expansions for increasing order m. We consider the approximations $\mathbf{u}^m$ from (2.5). Since the exact expansion is not available, we examine the solution differences at successive orders in the spirit of a Cauchy sequence. For the components u_l, the differences

$$E_l^m(t,x) := \|u_l^m(t,x,\xi) - u_l^{m-1}(t,x,\xi)\|_{L^2(\Omega)} = \left(v_{m,l}^m(t,x)^2 + \sum_{i=0}^{m-1} (v_{i,l}^m(t,x) - v_{i,l}^{m-1}(t,x))^2 \right)^{1/2} \tag{4.3}$$

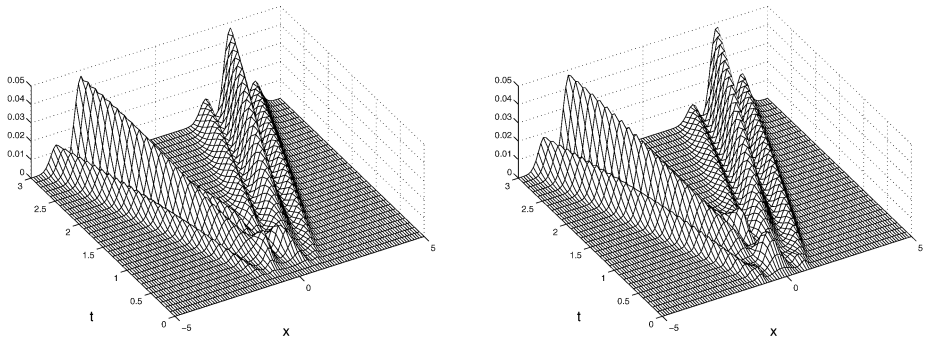


Fig. 5 Variance for u_1 (*left*) and u_2 (*right*) resulting from gPC for wave equation

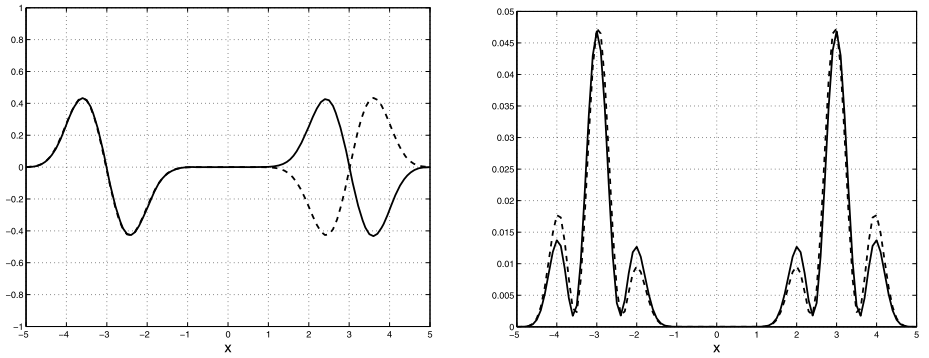


Fig. 6 Expected values (*left*) and variances (*right*) for u_1 (*solid line*) and u_2 (*dashed line*) at final time $t = 3$ resulting from gPC for wave equation

indicate the rate of convergence, where $v_{i,l}^m$ are the coefficient functions in $\mathbf{u}^m$. We solve the gPC systems for $m = 1, \ldots, 8$ to obtain the numerical solutions $\mathbf{u}^m$. Figure 8 shows the maximal differences (4.3) in the grid points for each component. We recognise an exponential convergence in the approximations, which is typical for the gPC approach. Moreover, the values for the two components coincide.

For comparison, we perform a quasi Monte-Carlo simulation using K samples ξ_k for the random parameter to achieve a reference solution. Thereby, we consider the exact solution of the initial-boundary value problem of the system (4.2) for each velocity $c(\xi_k)$. We compute the solutions of the larger coupled systems (2.6) in the gPC for different degrees m. Again the Lax-Wendroff method yields the approximations on a mesh with same sizes as above. We discuss the corresponding mean square errors

$$\bar{E}_l^m(t,x) := \left(\frac{1}{K} \sum_{k=1}^{K} \left(u_l(t,x,\xi_k) - u_l^m(t,x,\xi_k) \right)^2 \right)^{1/2} \tag{4.4}$$

for the components $l = 1, 2$. Table 1 illustrates the maximum mean square errors (4.4) on the grid using $K = 100$ and $K = 200$ samples. As expected, the differences decrease for increasing degree m, i.e., higher accuracy in the gPC. The results for different sample number K differ hardly, which indicates that the quasi Monte-Carlo simulation yields a sufficiently

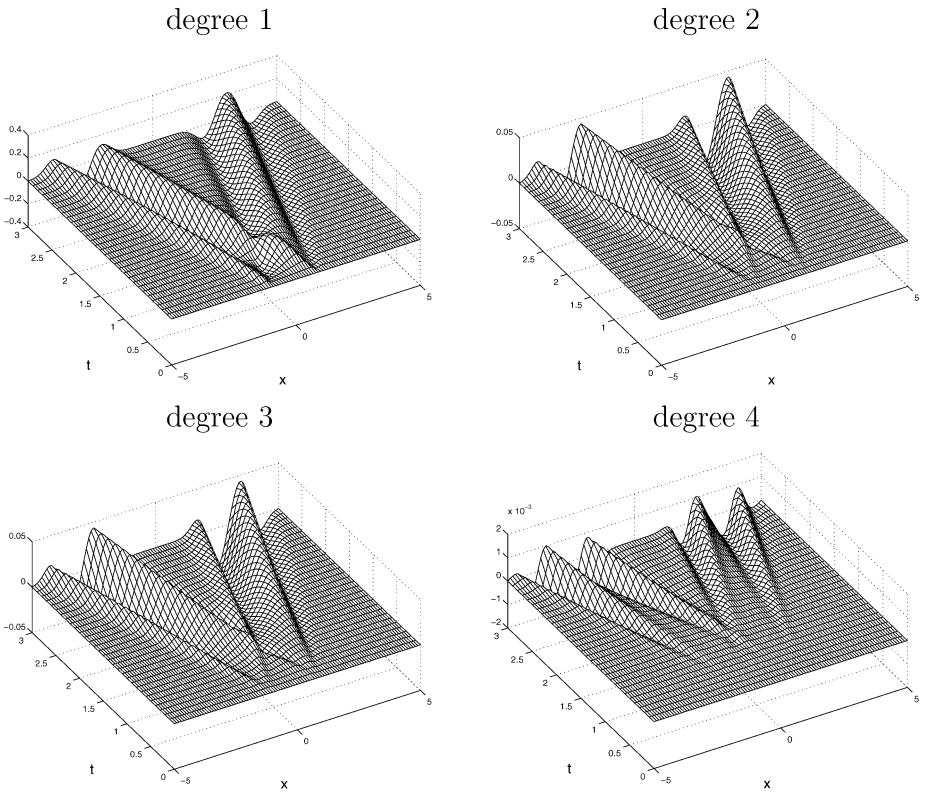


Fig. 7 Coefficient functions of u_1 in gPC for wave equation

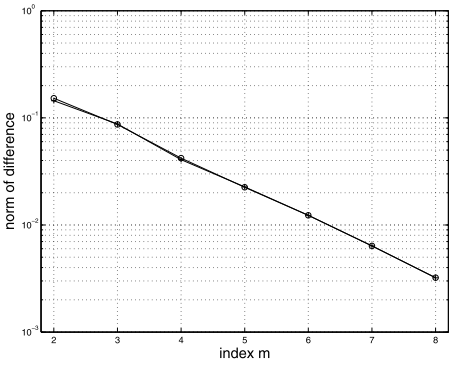


Fig. 8 Maximum of differences E_1^m (*circle*) and E_2^m (*cross*) in numerical solutions for different orders m from gPC in semi-logarithmic scale

accurate reference solution for comparing the gPC simulations. We remark that the mean square error decreases significantly if smaller step sizes are applied in space and time. Thus the error of the computed gPC solutions is dominated by the discretisation error in time and space and not by the error of the stochastic Galerkin approach.

Table 1 Maximum mean square errors between approximations from gPC for different degrees and quasi Monte-Carlo simulation with K samples

Degree m	$K = 100$		$K = 200$	
	u_1	u_2	u_1	u_2
1	$1.2 \cdot 10^{-1}$	$1.3 \cdot 10^{-1}$	$1.2 \cdot 10^{-1}$	$1.3 \cdot 10^{-1}$
2	$8.1 \cdot 10^{-2}$	$8.1 \cdot 10^{-2}$	$8.2 \cdot 10^{-2}$	$8.2 \cdot 10^{-2}$
3	$6.1 \cdot 10^{-2}$	$7.5 \cdot 10^{-2}$	$6.1 \cdot 10^{-2}$	$7.5 \cdot 10^{-2}$
4	$6.0 \cdot 10^{-2}$	$7.4 \cdot 10^{-2}$	$6.0 \cdot 10^{-2}$	$7.4 \cdot 10^{-2}$
5	$6.0 \cdot 10^{-2}$	$7.4 \cdot 10^{-2}$	$6.0 \cdot 10^{-2}$	$7.4 \cdot 10^{-2}$

4.2 Linearised Shallow Water Equations

The one-dimensional shallow water equations read

$$\frac{\partial}{\partial t}\begin{pmatrix} v \\ \varphi \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \frac{1}{2}v^2 + \varphi \\ v\varphi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with the water level $\varphi > 0$ and the velocity $v \in \mathbb{R}$, see [6]. The linearised shallow water equations are

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} \bar{v} & 1 \\ \bar{\varphi} & \bar{v} \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.5}$$

with constants $\bar{v}, \bar{\varphi}$. It follows that the linear system (4.5) is strictly hyperbolic for all $\bar{v} \in \mathbb{R}$ and all $\bar{\varphi} > 0$. We apply the constants $\bar{v} = 2, \bar{\varphi} = \frac{1}{2}$ and add random perturbations. We choose the matrix in the linear system (2.1) as

$$A(\xi_1, \xi_2) = \begin{pmatrix} 2 & 1 \\ \frac{1}{2} & 2 \end{pmatrix} + \xi_1\gamma\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \xi_2\gamma\begin{pmatrix} 0 & 0 \\ \frac{2}{5} & 0 \end{pmatrix} \tag{4.6}$$

with a Gaussian random variable ξ_1 ($\langle \xi_1 \rangle = 0$, $\langle \xi_1^2 \rangle = 1$) and a uniformly distributed random variable $\xi_2 \in [-1, 1]$. The constant $\gamma \in \mathbb{R}$ is used to control the magnitude of the variance in the random input later. It follows that the linear system (2.1) is strictly hyperbolic for each realisation of the random parameters provided that $|\gamma| < \frac{5}{4}$. We choose $\gamma = 1$ now.

The corresponding gPC approach applies products of the Hermite polynomials and the Legendre polynomials. We consider the two sets of basis polynomials $\mathcal{M}_R$ and $\mathcal{N}_R$, respectively, see Sect. 3.3. We calculate the matrix B in the coupled system (2.7) for different integers R. Thereby, Gaussian quadrature yields the probabilistic integrals in (2.6), where the results are exact except for roundoff errors.

Figure 9 illustrates the resulting eigenvalues of the matrix B from the coupled system (2.7) in case of $R = 2$ and $R = 10$. In both cases, pairs of complex conjugate eigenvalues occur for the basis $\mathcal{M}_R$. Hence the matrix B is not real diagonalisable. This counterexample demonstrates that the hyperbolicity of the larger coupled system (2.6) cannot be guaranteed in case of the basis $\mathcal{M}_R$. In contrast, the matrix B is real diagonalisable for the basis $\mathcal{N}_R$. This result is in agreement to Theorem 4.

Furthermore, Table 2 shows the total number of eigenvalues, which is equal to the order of the matrix, and the number of complex eigenvalues in case of the basis $\mathcal{M}_R$ for all $R = 1, \ldots, 10$. We recognise that the coupled system (2.7) is not hyperbolic for each $R > 1$. Thus an improvement of the accuracy in the gPC expansion does not result in a gain of hyperbolicity.

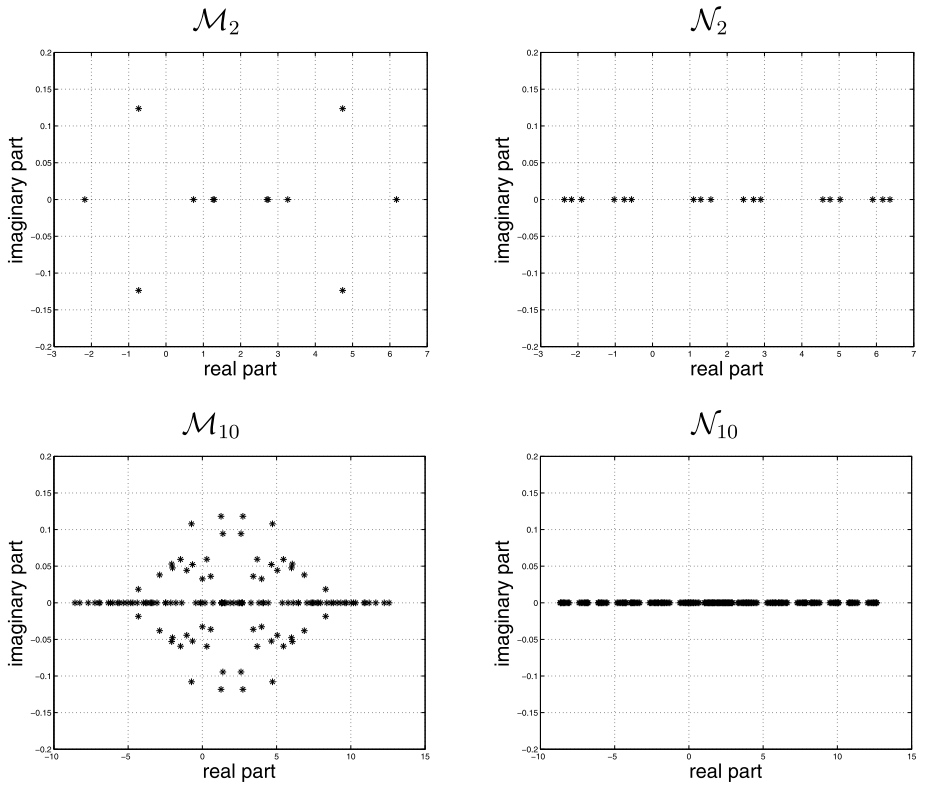


Fig. 9 Eigenvalues of matrix in coupled system for the linearised shallow water equations using different polynomial bases

Table 2 Total number of eigenvalues and number of complex eigenvalues for linearised shallow water equations in case of basis $\mathcal{M}_R$

R	1	2	3	4	5	6	7	8	9	10
Eig. val.	6	12	20	30	42	56	72	90	110	132
Comp. eig. val.	0	4	12	16	20	20	32	40	44	52

We also investigate the hyperbolicity in dependence on the magnitude of the variance of the random input parameters for the basis $\mathcal{M}_R$. The constant γ determines this variance due to (4.6). Figure 10 illustrates the maximum imaginary part in the spectrum of the matrix B from (2.7) using $R = 2$ as well as $R = 10$. The absence of complex eigenvalues is necessary for the hyperbolicity but not sufficient. Nevertheless, it has been checked numerically that a hyperbolic system (2.7) follows in case of real eigenvalues for our example. For all $R = 2, \ldots, 10$, the hyperbolicity is given for sufficiently small variance in the random input parameters. For larger variances, the hyperbolicity is lost and regained several times.

We perform a numerical simulation of the coupled system (2.6) with the polynomial basis $\mathcal{N}_3$ using $\gamma = 1$ in (4.6). Thus 16 basis functions appear and the order of B from (2.7) is 32. Since the solution of the linearised system (4.5) can be seen as a perturbation around

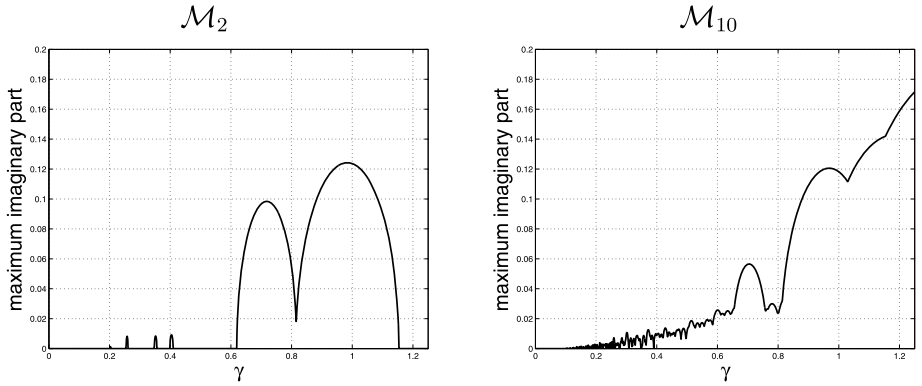


Fig. 10 Maximum imaginary part of eigenvalues of matrix in the coupled system using basis $\mathcal{M}_R$ for different magnitudes of stochastic input characterised by the constant $\gamma \in [0, \frac{5}{4}]$ from (4.6)

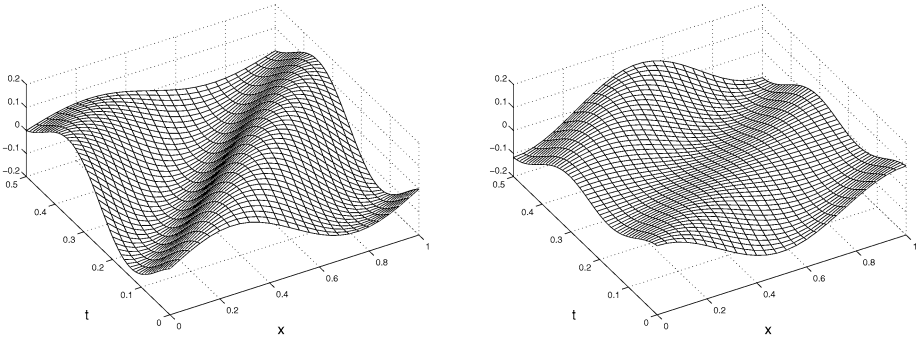


Fig. 11 Deterministic solutions u_1 (*left*) and u_2 (*right*) for linearised shallow water equations using the mean of the random parameters

the point of the linearisation, we consider the initial values

$$u_1(0,x) = \frac{1}{10}\sin(2\pi x), \qquad u_2(0,x) = \frac{1}{10}\cos(2\pi x).$$

We apply periodic boundary conditions for the space interval $x \in [0,1]$. Again the Lax-Wendroff scheme yields the numerical solution of the initial-boundary value problems of the linear PDE systems. We select the step sizes $\Delta x = \frac{1}{100}$ and $\Delta t = \frac{1}{1000}$, which satisfy the CFL condition. We calculate the solution in the time interval $t \in [0, \frac{1}{2}]$. For comparison, we solve the deterministic system (2.1) with the matrix (4.6) for $\xi_1 = \xi_2 = 0$, i.e., the expected values of the random parameters. Figure 11 illustrates the resulting deterministic solution. Figures 12 and 13 show the expected values and the variances, respectively, which follow from the gPC approach. In contrast to the previous test example, the expected values differ significantly from the deterministic solution using the mean values of the random parameters. Accordingly, the variances are relatively large, since the variances of the input parameters are higher than in the previous test example.

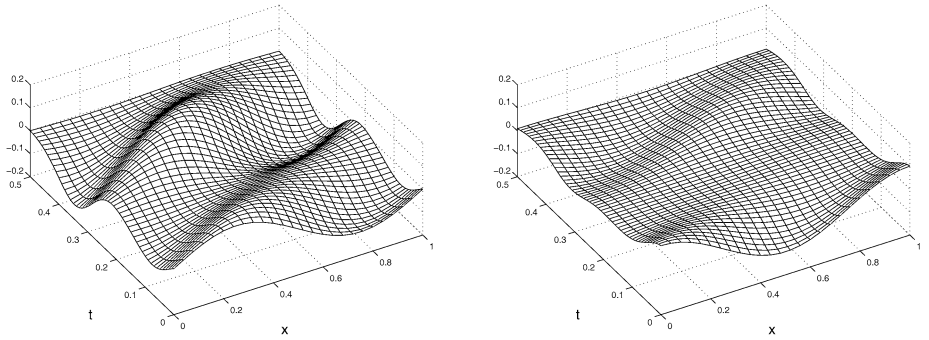


Fig. 12 Expected values for u_1 (*left*) and u_2 (*right*) resulting from gPC for linearised shallow water equations

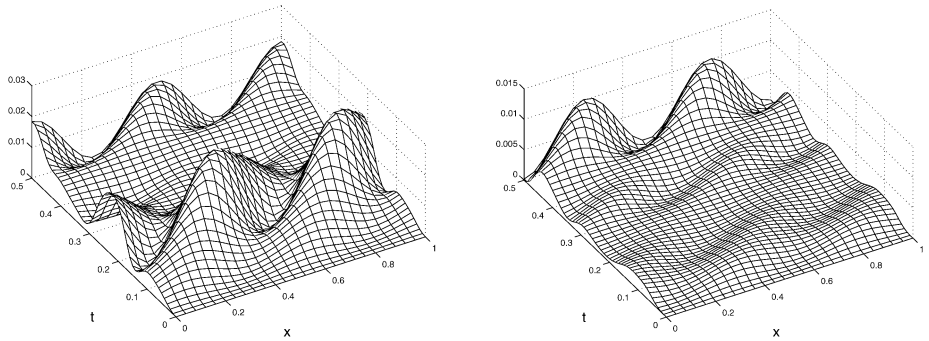


Fig. 13 Variance for u_1 (*left*) and u_2 (*right*) resulting from gPC for linearised shallow water equations

5 Conclusions

Linear conservation laws including random parameters can be resolved by the generalised polynomial chaos. Following the Galerkin approach, we obtain a larger coupled linear system of conservation laws for a certain class. Assuming that the original systems are hyperbolic, it follows that the coupled system is also hyperbolic in case of a single random parameter. Considering several random parameters, the hyperbolicity of the coupled system is not guaranteed for a basis of multivariate polynomials up to a fixed degree, which has been illustrated by a counterexample. In contrast, the hyperbolicity has been proved if a specific set of basis polynomials is applied, which exhibits a tensor product structure based on the univariate polynomials. Numerical simulations of two test examples illustrate that linear systems with random parameters can be solved successfully by the approach of the generalised polynomial chaos. The efficiency of this Galerkin approach is, however, a more complicated issue and must be understood on a problem-dependent basis.

Acknowledgements The author R. Pulch has been supported within the PostDoc programme of 'Fachgruppe Mathematik und Informatik' from Bergische Universität Wuppertal (Germany). The author D. Xiu has been supported by AFOSR FA9550-08-1-0353, NSF CAREER DMS-0645035, NSF IIS-091447, NSF IIS-1028291, DOE DE-SC0005713, and DOE/NNSA DE-FC52-08NA28617.

References

1. Augustin, F., Gilg, A., Paffrath, M., Rentrop, P., Wever, U.: Polynomial chaos for the approximation of uncertainties: chances and limits. Eur. J. Appl. Math. **19**, 149–190 (2008)
2. Chen, Q.-Y., Gottlieb, D., Hesthaven, J.S.: Uncertainty analysis for the steady-state flows in a dual throat nozzle. J. Comput. Phys. **204**, 387–398 (2005)
3. Constantine, P.G., Gleich, D.F., Iaccarino, G.: Spectral methods for parameterized matrix equations. SIAM J. Matrix Anal. Appl. **31**(5), 2681–2699 (2010)
4. Ghanem, R.G., Spanos, P.: Stochastic Finite Elements: A Spectral Approach. Springer, New York (1991)
5. Gottlieb, D., Xiu, D.: Galerkin method for wave equations with uncertain coefficients. Commun. Comput. Phys. **3**(2), 505–518 (2008)
6. LeVeque, R.J.: Numerical Methods for Conservation Laws. Birkhäuser, Basel (1990)
7. LeVeque, R.J.: Finite Volume Methods for Hyperbolic Problems. Cambridge University Press, Cambridge (2002)
8. Lin, G., Su, C.-H., Karniadakis, G.E.: Predicting shock dynamics in the presence of uncertainty. J. Comput. Phys. **217**, 260–276 (2006)
9. Poette, G., Despres, D., Lucor, D.: Uncertainty quantification for system of conservation laws. J. Comput. Phys. **228**, 2443–2467 (2009)
10. Pulch, R., van Emmerich, C.: Polynomial chaos for simulating random volatilities. Math. Comput. Simul. **80**(2), 245–255 (2009)
11. Pulch, R.: Polynomial chaos for linear differential algebraic equations with random parameters. Int. J. Uncertain. Quantific. **1**(3), 223–240 (2011)
12. Tryoen, J., Le Maître, O., Ndjinga, M., Ern, A.: Intrusive Galerkin methods with upwinding for uncertain nonlinear hyperbolic systems. J. Comput. Phys. **229**(18), 6485–6511 (2010)
13. Xiu, D.: The generalized (Wiener-Askey) polynomial chaos. Ph.D. thesis, Brown University (2004)
14. Xiu, D.: Fast numerical methods for stochastic computations: a review. Commun. Comput. Phys. **5**, 242–272 (2009)
15. Xiu, D.: Numerical Methods for Stochastic Computations: A Spectral Method Approach. Princeton University Press, Princeton (2010)
16. Xiu, D., Hesthaven, J.S.: High order collocation methods for differential equations with random inputs. SIAM J. Sci. Comput. **27**(3), 1118–1139 (2005)
17. Xiu, D., Karniadakis, G.E.: The Wiener-Askey polynomial chaos for stochastic differential equations. SIAM J. Sci. Comput. **24**(2), 619–644 (2002)