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Nonuniform Spatial Mesh Adaptation Using A Posteriori Error Estimates: Applications to Forward and Inverse Problems

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Abstract

In this paper we present an adaptive method for the solution of large scale forward and inverse problems occurring in biomedicine. Specifically, we describe models based on magnetic resonance images of the human thorax that are used to solve electric and potential field imaging problems. We introduce two- and three-dimensional nonuniform spatial mesh adaptation schemes based upon *a posteriori* error estimates from the finite element approximation, an algorithm that utilizes a minimax theorem based on estimates of the electric and potential fields and a three-dimensional, Delaunay triangulation program. Initial results are presented for two- and three-dimensional models.

Introduction

Over the last twenty-five years, researchers in biomedicine have utilized numerical techniques to solve a wide range of field problems that arise from the naturally occurring electric and magnetic fields generated by the body [1, 2, 3, 4]. The solution techniques usually center around classical numerical treatments to solve partial differential equations, finite difference, finite element, and boundary element methods. While biomedical researchers have evaluated [5, 6, 7] and compared these different numerical techniques [8] in animal experiments, the specific question of the choice of geometrical model resolution has not been addressed. In contrast to mechanical and aeronautical engineers, who have studied adaptive mesh strategies for more than a decade, bioengineers studying bioelectric field problems have been slow to investigate such methods. Instead they have typically assigned spatial discretization levels by relying largely on previous experience and common sense (limited by what could realistically be computed) without employing specific, quantitative methods to evaluate the adequacy of the mesh. For the most part, biomedical researchers have based their assumptions on simple problems for which analytic solutions are available. Mesh resolution was increased, usually uniformly, until the numerical solution approached that derived analytically and the same level of discretization was then applied to more complex and realistic geometries, with the belief that errors would remain approximately equivalent. It should be noted that the problems in biomedicine often require particularly complex geometrical models, including numerous inhomogeneous, even anisotropic regions so that computational limits have played a major role in restricting the fidelity of the models.

We will show that by using *a posteriori* estimates from the finite element approximation of the governing equations, along with a minimax theorem to determine the stopping criterion, we can not only effectively minimize the errors that affect the standard direct field solution, but also improve the accuracy of the associated inverse problem by providing a better representation of the corresponding transfer matrix.

The problems we will consider are fundamental to the field of theoretical electrocardiology. Briefly, there are three related problems of interest. The first is the direct problem: given a subset of potentials on the surface of the heart, or a description of the primary current sources within the heart, one must calculate the electric and potential fields everywhere within the body and the voltages on the surface of the torso. The second is the inverse of the first: given a subset of voltages on the surface of the thorax, one must calculate either the voltages on a closed surface bounding the heart or the primary currents within the volume of the heart. The last of the problems of interest is a variation of the first: given known currents or voltages which are applied from external sources (e.g. defibrillation electrodes), one must determine the distribution of applied current throughout the heart. Both of the inverse problems are ill-posed in the Hadamard sense, and the latter has the added disadvantage of having no unique solution. The third is a classic problem of external field stimulation. Note that in all of the above, one assumes *a priori* knowledge of the geometry of the thorax, including internal structures and their relative conductivities. This contrasts with the inverse problem of electrical impedance tomography, where one injects current into the body in order to determine the electrical conductivity of the internal tissues [9, 10]. In the problems we describe, the geometries are extracted using other imaging modalities (usually magnetic resonance imaging), and electric conductivity values from the literature are assigned to each of the tissue groups [11].

For all of the problems stated above, the ability to obtain accurate solutions depends on how well we can describe the geometry and conductivity values and control numerical errors, which occur in part because of the inadequate treatment of large potential gradients. Assuming for the moment that the physical geometry and conductivity are known, we are left with the problem of controlling numerical approximation errors due to inadequate discretization of the model geometry.

In this paper we address the issue of providing adequate mesh discretization using automatic adaptive methods. We begin with a mathematical outline of the problems and their solution using the finite element method. Then follows the development of an adaptive meshing scheme based on *a posteriori* error estimates of the finite element computations and a brief description of its implementation. Next we present and discuss some results using these techniques, and close with some speculations of the impact of these methods on the field of biomedical field computations.

Methods

Mathematically, the problems of theoretical electrocardiology can be stated as follows. First, find the electrostatic potentials Φ in Ω according to:

$$\nabla \cdot \sigma \nabla \Phi = -I_v \quad \text{in } \Omega \tag{1}$$

subject to the boundary conditions:

$$\Phi = \Phi_0 \text{ on } \Gamma_E \text{ and } \sigma \nabla \Phi \cdot \mathbf{n} = 0 \text{ on } \Gamma_T \tag{2}$$

where Φ and Φ_0 are electrostatic potentials, I_v are the primary cardiac current sources per unit volume, σ is the conductivity tensor, and Γ_E and Γ_T are the epicardial (heart) and torso (body) surface boundaries respectively, which bound the solution domain, Ω. Once the electrostatic potentials are known, one can calculate the current density $\mathbf{J}$ according to:

$$\mathbf{J} = -\sigma \nabla \Phi. \tag{3}$$

This is the direct problem. One form of the inverse problem is stated as solving (1) for the primary cardiac currents, I_v according to the boundary conditions,

$$\Phi = \Phi_0 \text{ on } \Sigma \subset \Gamma_T \text{ and } \sigma \nabla \Phi \cdot \mathbf{n} = 0 \text{ on } \Gamma_T. \tag{4}$$

As noted previously, the solution to this problem in terms of cardiac source descriptions is not unique unless additional constraints are provided. A common approach is to solve a model problem in which the heart is broken up into N subdomains with the electrical activity of each subdomain represented by a variable dipole moment. One then seeks to determine the magnitude and direction of each of the dipoles as functions of time. Another approach to the inverse problem involves enclosing the heart with a bounding surface, and, instead of solving the Poisson equation in (1) for the primary cardiac currents, solving Laplace's equation for the potentials on the closed surface just outside the heart (according to the same boundary conditions as in (4)) [12]. The advantages of this formulation include that the solution to the resulting inverse problem is unique and that these solutions can be directly verified by measuring the potentials on the outer surface of the heart [7]. While the solution to this problem is unique, it is still ill-posed in the Hadamard sense that small errors in input data can cause large unbounded oscillatory errors in the solution data. One usually has to implement some kind of regularization scheme to attempt to restore the continuity of the solution back onto the data.

Finally, the defibrillation problem can be posed as solving Laplace's equation with boundary conditions similar to those in (2), replacing *a prior* knowledge of voltage on the heart boundary by that on the region of the torso under the defibrillation electrodes, that is, $\Phi = \Phi_0$ on $\Sigma \subset \Gamma_T$. Here, the sources are due to defibrillation pulses and the goal is to determine the resulting current density throughout the heart volume.

In this paper, we shall concentrate on the solution of the direct and inverse problem in terms of voltages, solving Laplace's equation for either the potentials on the body surface (direct problem) or on the heart surface (inverse problem, the so-called Cauchy problem for Laplace's equation). The approximation methods thus far utilized to solve these problems include the finite difference, finite element, and boundary element methods (as of yet, neither multigrid nor other multi-level approaches have been described in the bioengineering literature). For the direct problem, we need to solve Laplace's equation in a complex, three-dimensional, inhomogeneous, anisotropic domain. Because of the anisotropy and the changing inhomogeneity, the finite element method is obviously the method of choice.

We start by considering the weak form of (1):

$$a(\Phi, \tilde{\Phi}) = -(I_v, \tilde{\Phi}) \qquad \forall\, \tilde{\Phi} \in V \tag{5}$$

where

$$a(\Phi, \tilde{\Phi}) = \int_\Omega \nabla\Phi \cdot \nabla\tilde{\Phi}\, d\Omega \tag{6}$$

and

$$(I_v, \tilde{\Phi}) = \int_\Omega I_v \tilde{\Phi} \, d\Omega \tag{7}$$

where $\tilde{\Phi}$ is an arbitrary test function in the space V, defined as:

$$V = \{\tilde{\Phi} : \tilde{\Phi} \text{ is continuous on } \Omega, \nabla\tilde{\Phi} \text{ is piecewise continuous on } \Omega\}. \tag{8}$$

We discretize the solution domain, Ω, into a set of non-overlapping tetrahedra, $T_h = K_1, \ldots, K_n$ (or correspondingly into triangles for two dimensional problems), and define a finite dimensional subspace, V_h

$$V_h = \{\tilde{\Phi} : \tilde{\Phi} \text{ is continuous on } \Omega,\ \tilde{\Phi}|_K \text{ is linear for K} \in T_h\}. \tag{9}$$

We then form a linear combination of basis functions such that the values agree at the nodes N_i (vertices)

$$\tilde{\Phi} = \sum_{i=1}^{n} \alpha_i \Psi_i \qquad \alpha_i = \tilde{\Phi}(N_i). \tag{10}$$

The finite element approximation of (1) can thus be stated as: find $\tilde{\Phi} \in V_h$ such that

$$a(\Phi_h, \tilde{\Phi}) = -(I_v, \tilde{\Phi}) \qquad \forall\, \tilde{\Phi} \in V_h. \tag{11}$$

If we now let,

$$\Phi_h = \sum_{i=1}^{n} \xi_i \Psi_i \quad \xi_i = \Phi_h(x_i) \tag{12}$$

we can recast the original problem as solving the linear system

$$A\xi = b \tag{13}$$

where $A = (a_{ij}) = (\sigma_{ij} \nabla\Psi_i \nabla\Psi_j)$, and $b = (-I_v, \Psi_i)$ are the global stiffness matrix and load vector, respectively, and ξ is the solution vector of potentials. Here, the global stiffness matrix contains all the geometry and conductivity information of the problem, while the load vector contains the source terms and changes therein due to the Dirichlet boundary condition. We note that A is symmetric, positive definite, and very sparse.

In order to solve the inverse problem, we must reformulate (13). If we partition A as:

$$\begin{pmatrix} A_{TT} & A_{TV} & A_{TE} \\ A_{VT} & A_{VV} & A_{VE} \\ A_{ET} & A_{EV} & A_{EE} \end{pmatrix} \begin{pmatrix} \xi_T \\ \xi_V \\ \xi_E \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{14}$$

subject to the Dirichlet boundary conditions, then we can rearrange the matrix to solve directly for the epicardial voltages in terms of the measure body surface voltages,

$$\xi_E = (A_{TV}A_{VV}^{-1}A_{VE} - A_{TE})^{-1}(A_{TT} - A_{TV}A_{VV}^{-1}A_{VT})\xi_T \tag{15}$$

or, more succinctly, $\xi_E = K^{-1}\xi_T$, where K is the transfer matrix of coefficients relating the measured torso voltages to the voltages on the heart (we note that one can solve the direct problem using this formulation as well, $\xi_T = K\xi_E$).

Unfortunately, K is ill-conditioned, and regularization techniques are necessary to restore continuity of the solution back onto the data. Briefly, the error between the approximate and true values is represented by an error vector, $\varepsilon = \{\delta_1, \ldots, \delta_n\}$. One wishes to find an approximate vector, $\Phi_{E\varepsilon}$ such that $\Phi_{E\varepsilon} \to \Phi_E$ as $\varepsilon \to 0$. Two primary errors are those due to discretization, $\|\tilde{K} - K\| \leq \delta_1$, and those in measuring the torso voltages, $\|\tilde{\Phi} - \Phi\| \leq \delta_2$. We can find an estimate for $\Phi_{E\varepsilon}$ by minimizing a generalized form of the Tikhonov functional:

$$M^{\alpha}[\Phi_E, \tilde{\Phi}_T, \tilde{K}] = \|K\Phi_E - \tilde{\Phi}\|^2_{\phi_T} + \alpha\|C(\Phi_E - \Phi_E^{'}\|^2_{\phi_E} \quad \alpha > 0 \tag{16}$$

in terms of the epicardial potentials,

$$\Phi^{\alpha}_{E\varepsilon} = [\tilde{K}^T\tilde{K} + \alpha C^T C]^{-1}[\tilde{K}^T\tilde{\Phi}_T + \alpha C^T C\Phi_E^{'}] \tag{17}$$

where $\tilde{K}$ is the approximation of the true transformation matrix, K, $\tilde{\Phi}_T$ are the measured torso potential values, $\Phi_E^{'}$ are *a priori* constraints placed on the epicardial potentials, C is a constraint matrix, and α is the regularization parameter. The regularization parameter is found by utilizing a function which estimates the errors, ε. The details of regularization are beyond the scope of this paper, but may be found in [13, 14, 12, 15, 16]. Suffice it to say that the regularization step inevitably exacerbates any error in the forward solution so that an accurate forward transfer matrix has even more bearing on the solution of the associated inverse problem than it does in the original direct problem.

Adaptive Methods

In terms of adaptive methods, we are fortunate that the governing equation is elliptic because the fundamental theory for adaptive methods is most sound for this class of problems. However, there are still no clear-cut solutions to the problem of implementation, especially for large three-dimensional problems. In fact, success of the theory often exceeds that of the implementation, and many error estimates, while theoretically sound, are impossible or impractical to put into action.

For electrocardiologic problems, we have the advantage of knowing *a priorily* where all the possible regions of high gradients will be — in the region near the sources (heart or defibrillation electrodes). However, it is still necessary to determine both the appropriate level of discretization and the means of implementing the adaptation automatically.

Mathematically, we can obtain an error estimator by starting from the weak formulation in (11). We can prove that $\Phi_h \in V_h$ is the best approximation of the exact solution Φ in the sense that

$$\|\nabla\Phi - \nabla\Phi_h\| \leq \|\nabla\Phi - \nabla\tilde{\Phi}\| \quad \forall\, \tilde{\Phi} \in V_h. \tag{18}$$

where

$$\|\nabla\tilde{\Phi}\| = a(\tilde{\Phi}, \tilde{\Phi})^{\frac{1}{2}}. \tag{19}$$

In particular, if we choose $\tilde{\Phi}$ to be the interpolant of Φ

$$\pi_h\Phi(N_i) = \Phi(N_i) \qquad i = 1, \ldots, M \qquad \pi_h\Phi \in V_h \tag{20}$$

with certain constraints on our elements, we can prove the following well know relationships

$$\|\nabla\Phi - \nabla\pi_h\Phi\| \leq Ch \tag{21}$$

and

$$\|\Phi - \pi_h\Phi\| \leq Ch^2, \tag{22}$$

where C is a positive constant that depends on the size of the second partial derivative of Φ and the smallest angle formed by the sides of the elements in V_h. To make this a little more precise, we define the semi-norm:

$$|\tilde{\Phi}|_{H^r(\Omega)} \;=\; (\sum_{|\alpha|=r} \int_\Omega |D^\alpha \tilde{\Phi}|^2 \, dx)^{\frac{1}{2}} \tag{23}$$

where

$$D^\alpha \tilde{\Phi} \;=\; \frac{\partial^{|\alpha|}\tilde{\Phi}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \qquad |\alpha| \;=\; \alpha_1 \;+\; \alpha_2 \tag{24}$$

are α-order partial derivatives and

$$H^K(\Omega) = \{\tilde{\Phi} \in L_2(\Omega) : \; D^\alpha \tilde{\Phi} \in L_2(\Omega), \; |\alpha| \le K\} \tag{25}$$

are the usual Sobolev spaces. Given these definitions, we can then obtain the following error estimates for the finite element approximation [17, 18]:

$$|\Phi - \Phi_h|_{H^1(\Omega)} \;\le\; |\Phi - \pi_h \Phi|_{H^1(\Omega)} \;\le\; C[\sum_{K \in T_n} (h_k |\Phi|_{H^2(K)})^2]^{\frac{1}{2}} \tag{26}$$

or, in the $L_2(\Omega)$ norm,

$$\|\Phi - \Phi_h\|_{L_2(\Omega)} \;\le\; \|\Phi - \pi_h \Phi\|_{L_2(\Omega)} \;\le\; C h^2 |\Phi|_{H^2(\Omega)}. \tag{27}$$

We can also provide error estimates for the potential gradients [19]

$$\|\nabla\Phi - \nabla\Phi_h\|_{L_\infty(\Omega)} \le C\|\nabla\Phi - \nabla\pi_h\tilde{\Phi}\|_{L_\infty(\Omega)} \le C\max[h_K \max_{\alpha=2} \|D^\alpha\Phi\|_{L_\infty(\Omega)}]. \tag{28}$$

We can now use the above estimates to create an adaptive algorithm. Suppose we want the accuracy of our finite element approximation to be within some tolerance, δ, of the true solution, i.e.

$$|\Phi - \Phi_h|_{H^1(\Omega)} \;\le\; \delta \tag{29}$$

then, from the estimates above, we have

$$\sum_{K \in T_h} (h_k |\Phi|_{H^2(K)})^2 \;\approx\; \frac{\delta^2}{C^2}. \tag{30}$$

Thus we redefine our mesh until,

$$\sum_{K \in T_h} (h_k |\Phi_h|_{H^2(K)})^2 \;\le\; \frac{\delta^2}{C^2} \tag{31}$$

and/or

$$h_K \max_{\alpha=2} \|D^\alpha \Phi\|_{L_\infty(\Omega)} \le \frac{\delta}{C}. \tag{32}$$

Here we sum over all the K elements in our tessellated geometry, T_h, and test to see if the error is less than our *normalized* tolerance. If the value of the error estimate exceeds the tolerance, then the element is "flagged" for further refinement.

Computation and Implementation

The error estimates given above have been implemented for two- and three-dimensional geometries, and linked to an automatic Delaunay triangulation algorithm. Because the Delaunay criterion seeks to construct optimal tetrahedra (triangles) from a given set of unstructured points [20, 21], we chose to utilize this mesh generation methodology instead of the traditional "divide and conquer" technique. We also wanted to compare convergence rates obtained by first adding points locally, then completely re-constructing a new optimal mesh to the more traditional method of successive subdivision of the elements during adaptation. While we have yet to realize an optimal convergence rate strategy, we have implemented an algorithm that optimizes a nonuniform mesh for accuracy.

Of significant practical importance is choosing a reasonable tolerance, δ. Since the location and bounds of the potential gradients are dictated by known physiological considerations, we can generate an initial mesh based upon simple field strength–distance arguments. To estimate the electric field distribution around the heart, we can use a simple model of concentric spheres (or cylinders) within a homogeneous volume conductor. In this model, the inner and outer spheres represent the heart and body surfaces respectively. We can then calculate voltages everywhere in the volume using physiologically bounded voltages as the Dirichlet boundary condition. This allows us to produce a graded initial mesh by *a priorily* estimating the potential gradients and adjusting the discretization level accordingly. By this calculation, we are trying to assure that the errors far from the sources are minimal. Once the initial mesh is in place, we first solve (13) (with $b = 0$) using the finite element method, then apply the solution to (31) and (32) to approximate the error in both the potential and electric fields. The derivatives in (31) and (32) are approximated by standard central difference quotients. Because we have calculated the initial mesh from physical (and physiological) considerations, we can chose δ to be some fraction of the estimated error generated by elements furthest from the sources as a suitable stopping criterion for the adaptive process. Our goal now is to reduce the error in some global sense to the level found in the distant elements.

We can accomplish this by using the global error estimators from above, defining ε_g as the maximum estimated error in the gradient

$$\varepsilon_g = \max_{i=1,\ldots,n} \varepsilon_{ig} \tag{33}$$

and ε_p as the maximum estimated error in the potential

$$\varepsilon_p = \max_{i=1,\ldots,n} \varepsilon_{ip} \tag{34}$$

and minimizing the total error according to

$$\min \varepsilon_\alpha = \min(\max_{i=1,\ldots,n} \varepsilon_{i\alpha}) \qquad \alpha = p \text{ or } g. \tag{35}$$

Optimality is obtained when

$$\varepsilon_{i\alpha} = \text{ a constant} \qquad i = 1,\ldots,n; \;\; \alpha = p, g \tag{36}$$

where the constant is determined from the minimum error estimate based on initial estimates from the distant elements.

Results and Discussion

We have completed full direct simulations from two- and three-dimensional models of the human thorax. The two-dimensional models correspond to a single triangulated MRI scan from the mid-thoracic region, as shown in figure (1). Each triangle has been assigned a conductivity tensor based upon its location relative to the bounding surfaces of the thorax, which consist of the torso, the fat/muscle interface, the interior boundary of the skeletal muscle, the lungs, the atrial region of the heart, and an epicardial fat pad. Any element outside all the other defined regions was assigned a bulk conductivity value. The Dirichlet boundary conditions consisted of 1) simulated sources in the form of simple bipoles, and 2) voltages (64 unipolar electrograms) recorded from a patient with Wolff-Parkinson-White (WPW) syndrome during open chest surgery. The simulated source calculations were performed by placing a bipole (two charges of opposite polarity separated by a finite distance, d) on the surface of the heart and utilizing the adaptive procedure described above to refine the mesh automatically.

Insert Figures 1-3 here

Figures 2a-2f depict the mesh structure progressing through five iterations of the mesh adaptation algorithm, while figures 3a-3f show simulation results in the form of isovoltage contours. Figure 4 shows a graph of voltage-versus-electrode number on the body surface (electrodes start at the anterior midline of the thorax and progress counterclockwise). One immediately notes the significant differences between the calculations based on the initial versus the final meshes. Plots of the estimated error in voltage and potential gradient, shown in figure 5, reveal that after only a few applications of the adaptation process, the error reached a plateau. A simulation utilizing heart potentials from a single instant in time is shown in figure 6. The figure shows the potential distribution on the most refined mesh. Note that for instances in the cardiac cycle where there were no high gradients, the initial mesh proved to be quite adequate. However, when there were significant spatial gradients, errors occurred on the same order as those in the bipole simulations.

Insert Figures 4-6 here

Given that for the inverse problem, small errors in the input data can cause large errors in the solution, one can easily anticipate, at least qualitatively, the effects of mesh discretization error on the inverse solution. Furthermore, since the transfer matrix of coefficients which relate body surface to epicardial surface potentials consists entirely of geometrical and conductivity information, even with a "perfect" regularization scheme, one would, at best, only be able to recover an inverse which was as good as the direct solution that produced it. It has always been assumed — and our findings support this notion — that improving the accuracy of the direct solution also improves the subsequent inverse solution. In our initial inverse studies, we utilized the two-dimensional model as described previously. The original mesh contained approximately 1500 triangular elements while the final refinement, after five iterations of the adaptive algorithm, contained approximately 7000 elements. The application of a local regularization technique and the adaptive algorithm recovered voltages to within 12.6% RMS error, [16]. Previous studies have reported the recovery of epicardial potentials with errors in the range of 20-40%, [15, 22].

Thus far, we have completed calculations for the direct problem using a single heartbeat (approximately 1 s of data) from a patient with WPW syndrome using a three-dimensional model of the human torso containing approximately 26,000 nodes and 150,000 tetrahedral elements which is a medium resolution version of the full model which contains over 82,000 nodes and over 575,000 elements, [23, 24, 25]. Due to the quasi-static nature of the problem, we were able to distribute the 500 different calculations (for data recorded from the heart surface at 2 ms intervals) over several workstations (IBM RISC-6000s, models 520, 530, and 550). Figure 7 shows results from a simulation using the inhomogeneous three-dimensional model. The torso on the right shows a rendered isovoltage map, while the torso on the left shows an interior view of the electric currents which flow in the volume. Figure 8 shows a closeup view of the currents near from the heart. A tetrahedralized section has been superimposed to reveal the discretization level.

Insert Figures 7-8 here

We have recently implemented the adaptive algorithms for the three-dimensional models which add new nodes at the centroids of the tetrahedra and along the bisector of any length which belongs to a boundary. The connectivities are broken and a new tetrahedralization is constructed according to the Delaunay criteria, much as in the two-dimensional case. One immediate drawback with the h methodology of mesh adaptation is the resulting increase in the number of degrees of freedom. While our current mesh is fairly coarse, computational limits require the local addition of nodes be accompanied by simultaneous removal of those that contribute little to the accuracy of the solution. Such a balanced scheme of local mesh adaptation is a topic of our current research. Specific issues include the management of mesh refinement with special data structures and finding the optimal balance between accuracy and calculation time.

There are many questions yet to be answered. One is the optimal way to introduce additional nodes into regions that must be subdivided. It is yet to be determined whether the Delaunay approach or the "successive subdivision" approach is the best (optimal) method for accomplishing this. Another issue to be addressed is the cost *vs.* benefit of implementing automatic adaptive schemes of the $h-p$ type, i.e., increasing both the spatial discretization level of the mesh and the order of the polynomial basis functions. While it seems intuitively obvious that some sort of $h-p$ method should be optimal, what is much less clear is how to decide whether a p or h step is appropriate, and how to find the optimal balance between the two especially in the context of an automatic algorithm. This issue becomes even cloudier for complex anisotropic inhomogeneities, where a higher order element might straddle two (or more) inhomogeneous, possibly anisotropic, regions. At present, we are investigating these possibilities with the goal of extending our initial results to the full three-dimensional direct and inverse problems.

Conclusion

This paper describes our initial studies in implementing an adaptive method for solving large scale three-dimensional field problems in biomedicine. We have developed an adaptive algorithm that utilizes *a posteriori* error estimates from the finite element approximation, a two and three-dimensional Delaunay mesh generator, and a physically (physiologically) based minimax condition to estimate the stopping criterion, for application to direct and inverse problems. Thus far, we have implemented the adaptive algorithms for two-dimensional direct and inverse problems, have initial results for three-dimensional direct problems, and are implementing the algorithms for the three-dimensional inverse problem (results for non-adaptive inverse solutions may be found in [14]). For problems in which experimental validation is difficult (or impossible) and the effects of errors are potentially catastrophic, as is frequently the case in medicine, confidence in the simulation results is crucial. Adaptive methods present an appealing way to decrease errors due to discretization and thus increase accuracy. While there are still many questions to be answered concerning the optimal implementation of such adaptive algorithms, we believe that we have provided a basis for further study in complex, large scale problems in medicine.

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Figure Captions

Figure 1. A triangulated magnetic resonance image. The colors correspond to different regions of electrical conductivity; yellow signifies the subcutaneous fat, red the skeletal muscle, purple the lungs, blue the lungs, and gray represents an average conductivity value.

Figure 2. Magnified views of the area occupied by the electric bipole. The original mesh is located in the upper left corner (panel A). The mesh was successively refined (panels B–F, running from left to right, from top to bottom, until the error fell below a predefined tolerance.

Figure 3. Shown here are the isovoltage contours from a simulation using the electric bipole. The meshes correspond to those in figure 2. Here, red contour line signify positive voltage, blue, negative voltage, and green values close to zero.

Figure 4. A graph of voltage versus electrode number. The voltages are calculated values at the body surface for the two-dimensional section shown in figures 1–3. Electrode numbers start at the anterior midline of the thorax (bottom center of figure 1) and progress counterclockwise.

Figure 5. A graph of the estimate error in the voltage and voltage gradient after each adaptation. Values shown are from the simulations using the same single bipole source as shown in figures 2 and 3.

Figure 6. The results of a direct simulation using recorded heart voltages as the Dirichlet boundary condition for the two-dimensional section. The results shown were calculated using the final mesh (after four adaptations). The color code for the contour lines is the same as that in figure 3.

Figure 7. Simulation using the inhomogeneous three-dimensional model. The boundary conditions were derived from 64 channels of unipolar electrograms recorded from the heart surface during open chest surgery. The torso on the right shows a color-coded, rendered isovoltage map, while the torso on the left has been opened to reveal the electric current loops which form in the thorax. The color map is the same as that in figure 3.

Figure 8. A magnified view of the currents near the heart. A tetrahedralized slice (in blue) has been superimposed around the heart (shown in red) to indicate the level of discretization. Color coding of the current lines is to indicate current density amplitude.