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Comments on the "Meshless Helmholtz-Hodge decomposition"

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Abstract—The Helmholtz-Hodge decomposition (HHD) is one of the fundamental theorems of fluids describing the decomposition of a flow field into its divergence-free, curl-free and harmonic components. Solving for an HDD is intimately connected to the choice of boundary conditions which determine the uniqueness and orthogonality of the decomposition. This article points out that one of the boundary conditions used in a recent paper "Meshless Helmholtz-Hodge decomposition" [5] is, in general, invalid and provides an analytical example demonstrating the problem. We hope that this clarification on the theory will foster further research in this area and prevent undue problems in applying and extending the original approach.

Index Terms—Vector Fields, Boundary Conditions, Helmholtz-Hodge Decomposition.

1 Introduction

The Helmholtz-Hodge decomposition (HHD) decomposes a vector field defined on a bounded or an unbounded domain into divergence-free, curl-free and harmonic components. Due to ubiquitous nature of vector fields, the HHD is used in a variety of applications in areas such as fluid modeling, computer graphics, computer vision, and topological analysis. In the case of bounded domains the boundary conditions play a crucial role in the decomposition, determine the existence, orthogonality, and the uniqueness of the decomposition.

In a "Meshless Helmholtz-Hodge decomposition" [5] Petronetto et al. propose a novel approach to compute the discrete HHD using smoothed particle hydrodynamics (SPH). As usual, they solve two Poisson equations to compute the HHD in this case based on a particle system framework. The meshless nature of this approach makes it attractive in many applications and the underlying framework is technically sound.

However, while surveying the HHD literature, a discrepancy in one of the boundary conditions proposed in [5] has emerged. In particular, Petronetto et al. suggest two sets of boundary conditions: The first are the traditional conditions found elsewhere in the field imposing a normal boundary flow on the curl-free and a tangential flow on the divergence-free component. The second set of boundary conditions is the inverse with normal flow in the divergence-free and tangential flow in the curl-free component. In this article, we point out that the second set of boundary conditions is invalid for general vector fields. Below, we provide an analytical example and show that the second set of boundary conditions violate the divergence theorem. For completeness, we first provide a brief description of the theory of HHD and the correct boundary conditions.

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2 The Helmholtz-Hodge decomposition

Consider a vector field $\vec{\xi}$ defined on a simply-connected, bounded domain $\Omega\left(\subset \mathbb{R}^2, \mathbb{R}^3\right)$ with boundary $\partial\Omega$. The divergence and curl of $\vec{\xi}$ are given by $\nabla \cdot \vec{\xi}$ and $\nabla \times \vec{\xi}$. According to the HHD, $\vec{\xi}$ can be decomposed into a curl-free (purely divergent) vector field $\vec{d}$, a divergence-free (purely rotational) vector field $\vec{r}$ and a harmonic vector field $\vec{h}$. Furthermore, the curl-free component can be represented as the gradient of a scalar potential function, i.e. $\vec{d} = \nabla D$, and the divergence-free component can be represented as the curl of a vector potential function, i.e. $\vec{r} = \nabla \times \vec{R}$.

$$\begin{aligned} \vec{\xi} &= \vec{d} + \vec{r} + \vec{h} \\ &= \nabla D + \nabla \times \vec{R} + \vec{h} \end{aligned} \tag{1}$$

Here, $\vec{d}$ represents all the divergence in $\vec{\xi}$, i.e. $\nabla\cdot\vec{\xi} = \nabla\cdot\vec{d}$, and $\vec{r}$ represents all the curl in $\vec{\xi}$, i.e. $\nabla \times \vec{\xi} = \nabla \times \vec{r}$.

Note, that for simply-connected domains, sometimes the rotation-free and the harmonic components are together represented as the gradient of a scalar field, i.e. $\vec{d} + \vec{h} = \nabla D$, which gives the two-component form of the Helmholtz-Hodge decomposition, e.g. as explained by Chorin and Marsden [1], Denaro [2] and Lamb [4]. However, when the domain is non-simply-connected, or when it is desirable to represent the harmonic as a separate component, the three-component form (Eq. 1) is used, e.g. Polthier and Preuß [6], [7], Tong et al. [8] and Petronetto et al. [5].

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For a unique and L^2-orthogonal decomposition, the commonly used boundary conditions are:

1) The curl-free component is normal to the boundary, i.e. $\vec{d} \times \vec{n} = 0$.
2) The divergence-free component is parallel to the boundary, i.e. $\vec{r} \cdot \vec{n} = 0$.

where, $\vec{n}$ is the outward normal to the boundary.

The proof of uniqueness and orthogonality for these boundary conditions can be found in [1], [2]. For shorthand, we call these the *normal-parallel (N-P)* boundary conditions. Most of the literature available in visualization and graphics and fluid modeling enforces the N-P boundary conditions for a unique decomposition. To compute HHD, first the potentials D and R are solved for, giving $\vec{d}$ and $\vec{r}$, and then the harmonic component is computed as the residual $\vec{h} = \vec{\xi} - \vec{d} - \vec{r}$.

HHD in 2D. For $\mathbb{R}^2$ or 2-manifolds embedded in $\mathbb{R}^3$, curl is just a scalar quantity normal to the manifold. Hence, the divergence-free component can be represented as the gradient of a scalar potential. Consider an operator J which rotates a vector counter-clockwise by $\pi/2$. Using J, it can be shown that

- $\nabla \times \vec{v} = -\nabla \cdot (J\vec{v})$
- $(J\vec{v}) \cdot \vec{n} = \vec{v} \perp \vec{n} =$ component of $\vec{v}$ parallel to the boundary, where $\vec{n}$ is the outward normal to the boundary.
- $\vec{v} = J\nabla V$ gives a divergence-free vector field, where V is a scalar field.

For details, we refer the reader to [7]. Thus, the 2D HHD can be written as

$$\vec{\xi} = \nabla D + J\nabla R + \vec{h} \tag{2}$$

where, R is a scalar potential. Now, for the curl-free component, $\nabla \times \vec{d} = \nabla \cdot \left(J\vec{d}\right) = 0$. By the divergence theorem and the properties of J,

$$\int_\Omega \nabla \cdot \left(J\vec{d}\right) \; dA = \int_{\partial\Omega} \left(J\vec{d}\right) \cdot \vec{n} \; dS$$
$$0 = \int_{\partial\Omega} \vec{d} \perp \vec{n} \; dS$$

Thus, the boundary condition $\vec{d} \times \vec{n} = 0$ can be replaced by $\vec{d} \perp \vec{n} = 0$ for an orthogonal decomposition.

3 THE BOUNDARY CONDITIONS IN [5]

In addition to the N-P boundary conditions, Petronetto et al. [5] show results for another set of boundary conditions for HHD in $\mathbb{R}^2$, which are opposite to the N-P boundary conditions. According to [5],

1) The curl-free component is parallel to the boundary, i.e. $\vec{d} \cdot \vec{n} = 0$.
2) The divergence-free component is normal to the boundary, i.e. $\vec{r} \perp \vec{n} = 0$.

We call these as the *parallel-normal (P-N)* boundary conditions, and provide an example demonstrating that in general the P-N boundary conditions are invalid.

Invalidity of the P-N boundary conditions. Consider a vector field defined by a source at the origin of $\mathbb{R}^2$. Such a field is given by $\vec{\xi} = (x, y)$. For $\vec{\xi}$, $\nabla \cdot \vec{\xi} = 2$, and $\nabla \times \vec{\xi} = 0$. Since $\vec{\xi}$ is irrotational, $\vec{r} = 0$. Also, since the field is purely divergent, $\vec{h} = 0$. Thus, computing the HHD for $\vec{\xi}$ should give $\vec{\xi} = \vec{d}$.

The divergence of $\vec{d}$ is simply $\nabla \cdot \vec{d} = \nabla \cdot \vec{\xi} = 2$. Combining the divergence theorem [3] and the P-N boundary conditions leads to,

$$\int_\Omega \nabla \cdot \vec{d} \; dA = \int_{\partial\Omega} \vec{d} \cdot \vec{n} \; dS \tag{3}$$
$$\Leftrightarrow \quad \int_\Omega 2 \; dA = \int_{\partial\Omega} 0 \; dS \tag{4}$$
$$\Leftrightarrow \quad 2\int_\Omega dA = 0 \tag{5}$$

Thus, for this example, a P-N style decomposition for $\vec{\xi}$ exists only for a domain with zero area. Essentially, by the divergence theorem, any P-N decomposition has zero global divergence and thus does not apply to most vector fields. More specifically, it can be applied only to the vector fields with $\int_\Omega \nabla \cdot \vec{\xi} \; dA = \int_{\partial\Omega} \vec{\xi} \cdot \vec{n} = 0$. Similarly, one can show that for a purely rotational field $\left(\vec{\xi} = (-y, x)\right)$, $\vec{r} \perp \vec{n} = 0$ cannot be maintained. The combined result is that P-N boundary conditions are not valid for general vector fields.

4 CONCLUSION

Boundary conditions determine the existence and mathematical properties of the HHD. Appropriate boundary conditions lead to a unique and L^2-orthogonal decomposition while improper boundary conditions can lead to an ill-posed problem. To avoid unnecessary complications when extending [5] we would like to caution the readers to avoid P-N boundary conditions for non-trivial boundary flow.

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