SIAM J. SCI. COMPUT.
Vol. 19, No. 6, pp. 2051–2060, November 1998

A SOLUTION-BASED TRIANGULAR AND TETRAHEDRAL MESH QUALITY INDICATOR*

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Abstract. A new mesh quality measure for triangular and tetrahedral meshes is presented. This mesh quality measure is based on both geometrical and solution information and is derived by considering the error when linear triangular and tetrahedral elements are used to approximate a quadratic function. The new measure is shown to be related to existing measures of mesh quality but with the advantage that local solution information in the form of scaled derivatives along edges is taken into account.

Key words. unstructured meshes, tetrahedral mesh quality, l_2 error information

AMS subject classifications. 65N30, 65N50

PII. S1064827596305222

1. Introduction.

The increasing use of PDE solvers based on triangular and tetrahedral meshes (e.g., [3], [12]) raises the issue of whether the mesh is appropriate to represent the solution. One approach to resolving this issue is have computable error estimates for each solution component. It is often the case that such estimates may not be available or may not be reliable. The usual approach is to view mesh quality as being independent of the solution [4, 6]. An alternative point of view is that it is both the shape of the elements and the local solution behavior that is important, particularly for highly directional flow problems [9, 10, 11]. The starting point for this work was the analysis of Babuška and Aziz [2], who showed that the requirement for triangles was that there should be no large angles. This work was extended to tetrahedral elements by Krizek [5] in a similar spirit.

The intention here is not to deal with the issue of how to construct an optimal mesh but instead to consider the related issue of how an existing mesh should be assessed given a solution. This reflects an important practical issue, particularly in three space dimensions, when a mesh generator produces a mesh of unknown quality for a complex solution. The requirement is then to assess how appropriate the mesh is for the computed solution. This paper is a step in this direction and will derive a simple mesh quality indicator, based on interpolation error estimates. The fundamental assumption being made is that the solution is being represented by a piecewise linear basis and that the function being approximated is quadratic. This assumption allows the error to be approximated by a quadratic function and the results of Nadler [7, 8] to be used for the triangular case. The resulting indicator will be shown to be related to those of Bank [3] and Weatherill, Marchant, and Hassan [12].

The second part of the paper will extend the work of Nadler to the case of a linear element tetrahedral mesh and so derive a new mesh quality indicator. This indicator will again be shown to behave in a similar way to that of Weatherill, Marchant, and Hassan [12] and its use will be illustrated by using parameterized examples from the work of Liu and Joe [6]. A parameterized tetrahedron combined with a simple model of a solution with highly directional gradients will be used to illustrate how the new

*Received by the editors June 12, 1996; accepted for publication (in revised form) March 10, 1997; published electronically July 27, 1998.
http://www.siam.org/journals/sisc/19-6/30522.html

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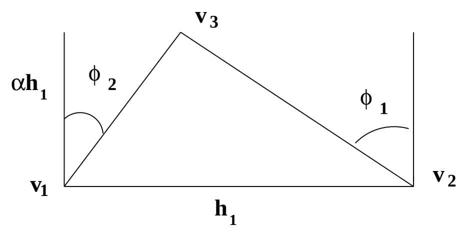

FIG. 1. *Example triangle.*

indicator indentifies the effect of directionality on the linear element approximation error.

2. Nadler's error estimate for triangles. The starting point for the derivation of a new mesh quality indicator is the work of Nadler [7] who derives a particularly appropriate expression for the interpolation error when a quadratic function is approximated by a piecewise linear function on a triangle. Consider the triangle T defined by the vertices v_1, v_2, and v_3, as shown in Figure 1. Let h_i be the length of the edge connecting v_i and v_{i+1} where $v_4 = v_1$. With reference to Figure 1, $h_2 = \alpha\ h_1\ \lambda_1$, $h_3 = \alpha h_1 \lambda_2$, $\lambda_1 = \sec(\phi_1)$, $\lambda_2 = \sec(\phi_2)$, $\mu_1 = \tan(\phi_1)$, and $\mu_2 = \tan(\phi_2)$.

Nadler [7] considers the case in which a quadratic function

$$u(x,y) = \underline{x}^T\ H\ \underline{x}, \quad \text{where} \quad \underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \tag{1}$$

is approximated by a linear function $u_{lin}(x,y)$, as defined by linear interpolation based on the values of u at the vertices. Denote the error by

$$e_{lin}(x,y) = u_{lin}(x,y) - u(x,y). \tag{2}$$

Nadler [7], as quoted in Rippa [11], shows that

$$\int_T (e_{lin}(x,y))^2 dx\ dy = \frac{A}{180}\left[(d_1+d_2+d_3)^2 + {d_1}^2 + d_2^2 + d_3^2\right], \tag{3}$$

where A is the area of the triangle and $d_i = \frac{1}{2}(v_{i+1}-v_i)^T\ H\ (v_{i+1}-v_i)$ is the derivative along the edge connecting v_i and v_{i+1}.

Example 1. In the case when the matrix H is positive definite with diagonal entries p^2 and q^2 and symmetric off-diagonal entries pq, then

$$d_i = (p\ \Delta x_i\ +\ q\ \Delta y_i)^2, \quad \text{where } v_{i+1} - v_i = [\Delta x_i, \Delta y_i]^T.$$

In the case of the triangle in Figure 1, assuming that x and y are in the horizontal and vertical directions, respectively, the values of d_i are $d_1 = p^2 h_1^2$, $d_2 = \alpha^2 h_1^2 (p\ \mu_1 + q)^2$, and $d_3 = \alpha^2 h_1^2 (p\ \mu_2 + q)^2$.

Example 2. In contrast, when the matrix H has diagonal entries p and p and symmetric off-diagonal entries q, then the matrix H has eigenvalues $p+q$ and $p-q$ and so is positive definite if $p > q$. In the case of the triangle in Figure 1, assuming that x and y are in the horizontal and vertical directions, respectively, the values of d_i for this matrix are

$$d_1 = p h_1^2\ ,\ d_2 = \alpha^2 h_1^2 (p(1+\mu_1^2) - 2\mu_1 q)$$

$$\text{and} \quad d_3 = \alpha^2 h_1^2 (p(1+\mu_2^2) - 2\mu_2 q)\ .$$

In this case d_2 and d_3 can be negative if both p and q are positive and $q \gg p$. It is also possible to pick α and μ_1 so that $d_1 + d_2 + d_3 = 0$ in this case and hence to zero part of equation (3).

3. A mesh quality indicator for linear triangular elements. In this section a new mesh quality indicator based on the work of Nadler [7] will be derived. This indicator takes into account both the geometry and the solution behavior. The starting point for this indicator is equation (4): in the case when the values of d_i are all equal then each edge makes an equal contribution to the error. However, in order to take into account in a consistent way the fact that the values of d_i may be of different signs it is necessary to consider their absolute values. It should also be noticed that if $d_i = h_i$ then the form of equation (3) has some similarities with the indicators of Bank [3] and Weatherill, Marchant, and Hassan [12]. This relationship will be made precise below. With these two points in mind, the scaled forms of the derivatives d_i are defined by

$$\tilde{d}_i = \frac{|d_i|}{d_{max}}, \quad \text{where } d_{max} \;=\; \max\left[|d_1|, |d_2|, |d_3|\right]. \tag{4}$$

For notational convenience define

$$\tilde{q}(\underline{\tilde{d}}) \;=\; (\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3)^2 + \tilde{d}_1^2 + \tilde{d}_2^2 + \tilde{d}_3^2, \tag{5}$$

where $\underline{\tilde{d}} = [\tilde{d}_1, \tilde{d}_2, \tilde{d}_3]^T$. A measure of the anisotropy in the derivative contributions to the error is then provided by

$$q_{aniso} \;=\; \frac{\tilde{q}(\underline{\tilde{d}})}{12}. \tag{6}$$

The definitions of the coefficients $\tilde{d}_i$ in equation (4) results in the bounds

$$0 \;\leq\; q_{aniso} \;\leq\; 1. \tag{7}$$

Consider a triangle with only one edge contributing to the error. In this case $q_{aniso} = 1/6$, whereas if two edges contribute equally and the third makes no contribution, $q_{aniso} = 1/2$.

In order to derive a consistent and related but geometry-only-based indicator it should be observed that the quantity defined by

$$q_m(\underline{h}) = \frac{\tilde{q}(\underline{h})}{16\;\sqrt{3}\;A},$$

where $\underline{h} = [h_1, h_2, h_3]^T$, has value 1 for an equilateral triangle and tends to the value infinity as the area of a triangle tends to zero but at least one of its sides is constant.

It is now possible to explain the relationship between this indicator and those of Bank [3] and Weatherill, Marchant, and Hassan [12] as denoted by q_b and q_w and defined by

$$\frac{1}{q_b} \;=\; \frac{1}{4\;\sqrt{3}\;A}\left[(h_1^2 + h_2^2 + h_3^2)\right], \qquad q_w \;=\; \frac{1}{3\;A}\left[(h_1 + h_2 + h_3)^2\right], \tag{8}$$

respectively. Hence

$$q_m(\underline{h}) = \frac{1}{4\;q_b} \;+\; q_w\,\frac{\sqrt{3}}{16}. \tag{9}$$

The relationship between q_{aniso} and the linear interpolation error is that when the matrix H is positive definite, i.e., $d_i > 0$, then

$$q_{aniso} = \frac{15}{A\ d_{max}^2} \int_T (e_{lin}(x,y))^2 dx\ dy, \tag{10}$$

thus showing that the indicator is a scaled form of the interpolation error in this special case.

3.1. Edge indices. In the case when q_{aniso} is small, it is possible to define an edge index which indicates how much each edge contributes to the error. Suppose that in equation (5) all the values of the terms $\tilde{d}_i$ are identical, say, $\tilde{d}_{avg}$; then

$$\tilde{q}(\tilde{d}) \;=\; 12(\tilde{d}_{avg})^2. \tag{11}$$

Hence

$$\tilde{d}_{avg} \;=\; \sqrt{q_{aniso}}. \tag{12}$$

The edge index for each edge is then denoted by $e_{ind}(i)$ and defined by

$$e_{ind}(i) \;=\; \frac{\tilde{d}_i}{\tilde{d}_{avg}}, \quad i = 1, 2, 3. \tag{13}$$

It is now possible to compare the approach adopted here with the recent mesh movement method of Ait-Ali-Yahia et al. [1], in which the H matrix is modified to be positive definite, and edge indicators, defined in the notation used here by $d_i/\sqrt{\Delta x_i^2 + \Delta y_i^2}$, are used to move the mesh. This approach thus scales the edge error component by the edge length. Ait-Ali-Yahia et al. [1] interpret d_i as the edge length in the H norm. The scaling defined by equation (13), in contrast, scales $|d_i|$ by an averaging factor taken over all the edges in the triangle. In the case when H is not positive definite, as in Example 2 of section 2, if the original values of d_2 and d_3 are negative (i.e., $q \gg p$) then the effect of the approach of [1] is transpose q and p in the H matrix and hence in the definitions of d_1, d_2, and d_3, thus giving different values from those in section 2:

$$d_1 = qh_1^2\ ,\ d_2 = \alpha^2 h_1^2 (q(1+\mu_1^2) - 2\mu_1 p),$$

$$\text{and} \quad d_3 = \alpha^2 h_1^2 (q(1+\mu_2^2) - 2\mu_2 p)\ .$$

3.2. Boundary layer flow example. The performance of this indicator may be illustrated by considering anisotropic flow, such as that in a viscous boundary layer, in which the two triangles defined as Case (a), Case (b), and Case (c) in Figure 2 are used to model a flow with a weak horizontal component $u_{xx} = 1$, an intermediate cross derivative $u_{xy} = 100$, and a strong vertical component $u_{yy} = 10000$. Case (a) is representative of a triangle thought to be especially suitable for such flows, while Case (b) is closer to the type of triangles produced by unstructured mesh generators. Table 1 shows the values of q_{aniso} for the three triangles as the height of the triangles α is varied. Also shown is the ratio of the L_2 errors for Case (a) and Case (b) divided by the error in Case (c). The table shows that in the case when $\alpha < 0.04$ triangles such as that in Case (c) are best. These results are explained by the indicator values and the values of d_{max}, which are $(1+100\alpha)^2$, $(0.5+100\alpha)^2$, and $(1+50\alpha)^2$ for cases (a), (b), and (c), respectively. For very small values of α, anisotropy is not a key factor, as the effective dominant flow direction is the horizontal one.

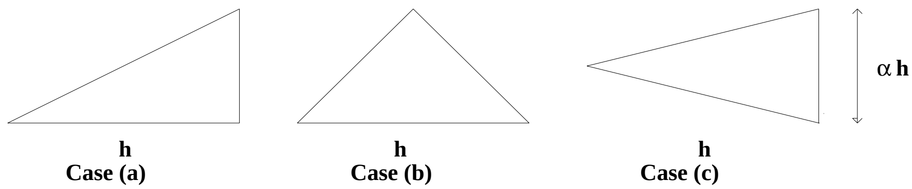

FIG. 2. *Boundary layer flow example triangles.*

TABLE 1
Mesh quality indicator values.

α	Case (a)	Case (b)	Case (c)	Error ratio a/c	Error ratio b/c
1.0	0.49	0.49	0.29	1.8	1.70
0.1	0.42	0.42	0.35	1.8	1.40
0.038	0.35	0.34	0.53	1.7	1.00
0.02	0.30	0.29	1.00	1.5	0.71
0.01	0.28	0.30	0.68	1.3	0.44
0.001	0.42	0.29	0.50	1.0	0.47
0.0001	0.49	0.28	0.50	1.0	0.55

4. Linear tetrahedral approximation of a quadratic function. The extension of Nadler's [7] approach to tetrahedra is achieved by considering the case in which a quadratic function

$$u(x,y,z) \;=\; \frac{1}{2}\,\underline{x}^T\,H\,\underline{x} \quad \text{where} \quad \underline{x} \;=\; \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{14}$$

is approximated by a linear function $u_{lin}(x,y,z)$ defined by linear interpolation based on the values of u at the vertices of a tetrahedron T defined by the vertices v_1, v_2, v_3, and v_4 as shown in Figure 3.

Let h_i be the length of the edge connecting v_i and v_{i+1} where $v_5 = v_1$. With reference to Figure 3 define the vectors $\hat{\underline{x}}$, $\hat{\underline{y}}$, $\hat{\underline{z}}$, $\hat{\underline{u}}$, $\hat{\underline{v}}$, and $\hat{\underline{w}}$ by

$$v_2 = v_1 + \hat{\underline{x}}, \quad v_3 = v_2 + \hat{\underline{y}}, \quad v_1 = v_3 + \hat{\underline{z}},$$

$$v_4 = v_1 - \hat{\underline{v}}, \quad v_4 = v_2 + \hat{\underline{w}}, \quad v_4 = v_3 + \hat{\underline{u}}, \tag{15}$$

and consequently

$$\hat{\underline{x}} + \hat{\underline{y}} + \hat{\underline{z}} \;=\; \hat{\underline{x}} + \hat{\underline{w}} + \hat{\underline{v}} \;=\; \hat{\underline{u}} + \hat{\underline{v}} - \hat{\underline{z}} \;=\; 0. \tag{16}$$

Define a reference tetrahedron T_{ref} (see Figure 3) by the four nodal points:

$$v_1 = (0,0,0)^T, \quad v_2 = (1,0,0)^T, \quad v_3 = (0,1,0)^T, \quad v_4 = (0,0,1)^T. \tag{17}$$

Then the mapping from the tetrahedron T_{ref} to the tetrahedron T is given by

$$\underline{x} = v_1 \;+\; B\,\tilde{\underline{x}}, \tag{18}$$

where $B \;=\; [\hat{\underline{x}}, -\hat{\underline{z}}, -\hat{\underline{v}}]$ and $\tilde{\underline{x}}$ is in the reference tetrahedron. T_{ref} and $\underline{x}$ is the equivalent point in the original tetrahedron T .

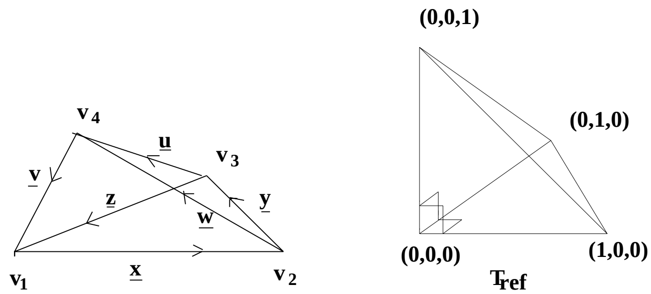

FIG. 3. *Example tetrahedron and reference tetrahedron.*

The function u may then be expressed as

$$u(x,y,z) \;=\; \frac{1}{2}\, v_1^T \; H \; v_1 \;+\; \frac{1}{2}\, \underline{\tilde{x}}^T \; B^T \; H \; v_1 \;+\; \frac{1}{2}\, v_1^T \; H \; B \; \underline{\tilde{x}} \;+\; \frac{1}{2}\, \underline{\tilde{x}}^T \; B^T \; H \; B \; \underline{\tilde{x}}, \qquad \text{where} \quad \underline{\tilde{x}} \;=\; \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{19}$$

is defined on T_{ref}. Ignoring the constant and linear terms (which are approximated exactly by a linear interpolant) and expanding the remaining quadratic term using equation (18) gives

$$\begin{aligned} u(x,y,z) \;=\; \frac{1}{2}\, & [(\underline{\hat{x}}^T H \underline{\hat{x}})x^2 + (-\underline{\hat{x}}^T H \underline{\hat{z}})2xy + (\underline{\hat{z}}^T H \underline{\hat{z}})y^2 \\ & (-\underline{\hat{x}}^T H \underline{\hat{v}})2xz + (\underline{\hat{z}}^T H \underline{\hat{v}})2zy + (\underline{\hat{v}}^T H \underline{\hat{v}})z^2]. \end{aligned}$$

Interpolating this by a linear function defined on T_{ref} by the nodal solution values gives

$$u_{lin}(x,y,z) \;=\; \frac{1}{2}\, \left[(\underline{\hat{x}}^T H \underline{\hat{x}})x + (\underline{\hat{z}}^T H \underline{\hat{z}})y + (\underline{\hat{v}}^T H \underline{\hat{v}})z \right], \tag{20}$$

and hence the linear interpolation error may be defined as:

$$e_{lin}(x,y,z) = u_{lin}(x,y,z) \;-\; u(x,y,z) \tag{21}$$

and written as

$$\begin{aligned} e_{lin}(x,y,z) = \frac{1}{2}\, & [(\underline{\hat{x}}^T H \underline{\hat{x}})(x - x^2) \;-\; (-\underline{\hat{x}}^T H \underline{\hat{z}})2xy \;+\; (\underline{\hat{z}}^T H \underline{\hat{z}})(y - y^2) \\ & -\; (-\underline{\hat{x}}^T H \underline{\hat{v}})2xz \;-\; (\underline{\hat{z}}^T H \underline{\hat{v}})2zy \;+\; (\underline{\hat{v}}^T H \underline{\hat{v}})(z - z^2)]. \end{aligned}$$

This in turn may be written as

$$e_{lin}(x,y,z) = \frac{1}{2}\, \underline{W}^T \; \underline{\hat{d}}, \tag{22}$$

where

$$\underline{W}^T = \left[x - x^2, \;\; -2xy, \;\; y - y^2, \;\; -2xz, \;\; -2zy, \;\; z - z^2\right]$$

and

$$\underline{\hat{d}}^T = \left[\underline{\hat{x}}^T H \underline{\hat{x}}, -\underline{\hat{x}}^T H \underline{\hat{z}}, \underline{\hat{z}}^T H \underline{\hat{z}}, -\underline{\hat{x}}^T H \underline{\hat{v}}, \underline{\hat{z}}^T H \underline{\hat{v}}, \underline{\hat{v}}^T H \underline{\hat{v}}\right].$$

Hence, from equation (22),

$$\int_T (e_{lin}(x,y,z))^2 dx\ dy\ dz = \frac{6V}{4} \int_{T_{ref}} \underline{\hat{d}}^T\ \underline{W}\ \underline{W}^T\ \underline{\hat{d}}\ \ dx\ dy\ dz, \tag{23}$$

where V is the volume of the tetrahedron. This may then be written as

$$\int_T (e_{lin}(x,y,z))^2 dx\ dy\ dz = \frac{6V}{4}\ \underline{\hat{d}}^T\ M\ \underline{\hat{d}}, \tag{24}$$

where the components $[M]_{ij}$ of the matrix M are defined in terms of the integrals of the ith and jth components of the vector $\underline{W}$ on the reference tetrahedral by

$$[M]_{ij} = \int_{T_{ref}} [\underline{W}]_i\ [\underline{W}]_j\ dx\ dy\ dz, \quad i,j = 1,\ldots,6.$$

A straightforward but lengthy calculation gives

$$M = \frac{2}{7!}\begin{bmatrix} 12 & -8 & 9 & -8 & -5 & 9 \\ -8 & 8 & -8 & 4 & 4 & -5 \\ 9 & -8 & 12 & -5 & -8 & 9 \\ -8 & 4 & -5 & 8 & 4 & -8 \\ -5 & 4 & -8 & 4 & 8 & -8 \\ 9 & -5 & 9 & -8 & -8 & 12 \end{bmatrix}.$$

It is now necessary to express the vector $\underline{\tilde{d}}$ in terms of the vector of second directional derivatives along edges defined by

$$\underline{d}^T = \frac{1}{2}\left[\underline{\hat{x}}^T H \underline{\hat{x}},\ \ \underline{\hat{y}}^T H \underline{\hat{y}},\ \ \underline{\hat{z}}^T H \underline{\hat{z}},\ \ \underline{\hat{u}}^T H \underline{\hat{u}},\ \ \underline{\hat{v}}^T H \underline{\hat{v}},\ \ \underline{\hat{w}}^T H \underline{\hat{w}}\right].$$

This is achieved by use of the vector identities defined by equation (16). For instance,

$$\underline{\hat{y}}^T H \underline{\hat{y}} = (\underline{\hat{x}} + \underline{\hat{z}})^T\ H\ (\underline{\hat{x}} + \underline{\hat{z}}),$$

and on expanding the right-hand side of this we get

$$-\underline{\hat{x}}^T\ H\ \underline{\hat{z}} = \frac{1}{2}\underline{\hat{x}}^T\ H\ \underline{\hat{x}} + \frac{1}{2}\underline{\hat{z}}\ H\ \underline{\hat{z}} - \underline{\hat{y}}^T\ H\ \underline{\hat{y}}.$$

A similar approach leads to the identities

$$-\underline{\hat{x}}^T\ H\ \underline{\hat{v}} = \frac{1}{2}(-\underline{\hat{w}}^T\ H\ \underline{\hat{w}} + \underline{\hat{x}}\ H\ \underline{\hat{x}} + \underline{\hat{v}}^T\ H\ \underline{\hat{v}})$$

and

$$\underline{\hat{z}}^T H \underline{\hat{v}} = \frac{1}{2}(\underline{\hat{v}}^T H \underline{\hat{v}}\ -\ \underline{\hat{u}} H \underline{\hat{u}}\ +\ \underline{\hat{z}}^T H \underline{\hat{z}}).$$

From these identities and the definitions of the vectors $\underline{\tilde{d}}$ and $\underline{d}$ it follows that

$$\underline{\hat{d}} \;=\; N\;\underline{d}, \quad \text{where} \;\; N = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix}.$$

Using this to substitute for $\underline{\hat{d}}$ in equation (24) gives

$$\int_T (e_{lin}(x,y))^2 dx\; dy\; dz = \frac{3}{2}\, V\; \underline{d}^T\; N^T\; M\; N\; \underline{d}\;\; . \tag{25}$$

Define the matrix P by

$$P \;=\; N^T\; M\; N \;\; = \frac{4}{7!} \begin{bmatrix} 4 & 2 & 2 & 1 & 2 & 2 \\ 2 & 4 & 2 & 2 & 1 & 2 \\ 2 & 2 & 4 & 2 & 2 & 1 \\ 1 & 2 & 2 & 4 & 2 & 2 \\ 2 & 1 & 2 & 2 & 4 & 2 \\ 2 & 2 & 1 & 2 & 2 & 4 \end{bmatrix}$$

and expand out equation (25) in terms of the components of $\underline{d}$ which are the six directional derivatives along the edges to get

$$\int_T (e_{lin}(x,y,z))^2 dx\; dy\; dz = \frac{6}{4}\; V\; \frac{8}{7!}\; \left[(\Sigma d_i)^2 - d_1 d_4 - d_2 d_5 - d_3 d_6 + \Sigma d_i^2\;\right]. \tag{26}$$

5. Tetrahedral mesh quality indicator. The results in the previous section make it possible to define the mesh quality indicator in the same way as in section 2, in that the error is scaled by the maximum directional derivative d_{max} and the integral is scaled by the volume before taking the square root. As in section 3, define

$$\tilde{Q}(\underline{\tilde{d}}) \;\; = \;\; \left[(\Sigma \tilde{d}_i)^2 - \tilde{d}_1 \tilde{d}_4 - \tilde{d}_2 \tilde{d}_5 - \tilde{d}_3 \tilde{d}_6 + \Sigma \tilde{d}_i^{\;2}\;\right], \tag{27}$$

where now $\underline{\tilde{d}} = [\tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \tilde{d}_4, \tilde{d}_5, \tilde{d}_6]^T$. As in section 3, a measure of the anisotropy in the derivative contributions to the error is then provided by

$$Q_{aniso} \;\; = \;\; \frac{\tilde{Q}(\underline{\tilde{d}})}{39}. \tag{28}$$

Again, as in section 3 and defining the normalized derivatives as in equation (3), a geometry-based indicator can be written as

$$Q_m(\underline{h}) = \frac{C}{V}\;\left[\;\tilde{Q}(\underline{\tilde{h}})\right]^{\frac{3}{2}}, \tag{29}$$

where C is a scaling factor to ensure that the indicator has value one when $h_i = h$ and thus $C = 1/(8.48528\;\; 39^{1.5})$ and the power of $\frac{3}{2}$ reflects the different dimensions of the error and the volume in powers of h.

The edge quality estimator used by Weatherill, Marchant, and Hassan [12] is of the form

$$Q_w = \frac{1}{8.48528V}\;\left[\left(\Sigma \frac{h_i}{6}\right)^3\;\right]. \tag{30}$$

TABLE 2
Mesh quality indicator values.

Tet. no.	4	5	6	7	8	9	10	11
I_{avg}	1.10	0.76	0.72	4.73	1.93	5.61	5.61	5.00
I_{max}	2.43	1.46	1.94	11.9	7.05	15.8	25.9	25.9

5.1. Edge indices. As in two dimensions it is possible to define an edge index which indicates how much each edge contributes to the error. Suppose that in equation (27) all the values of $\tilde{d}_i$ are identical, say, $\tilde{d}_{avg}$; then

$$\tilde{Q}(\underline{\tilde{d}}) = 39(\tilde{d}_{avg})^2. \tag{31}$$

Hence

$$\tilde{d}_{avg} = \sqrt{Q_{aniso}}. \tag{32}$$

The edge index for each edge is then denoted by $e_{ind}(i)$ and defined by

$$e_{ind}(i) = \frac{\tilde{d}_i}{\tilde{d}_{avg}}, \quad i = 1, 2, 3. \tag{33}$$

5.2. Numerical experiments. Before considering the performance of the indicator with nonuniform spatial gradients it is important to assess its performance on tetrahedra with uniform gradients. This may be done by using the eight parameterized tetrahedra of Liu and Joe [6] as defined by Figures 4–11 of that paper and defined here as Tetrahedra 4–11 for consistency. Liu and Joe's parameterization involves a constant u in the range $[0, 1]$, the value 1 representing a uniform tetrahedron, and the value zero a degenerate tetrahedron. For values of $u_n = 0.01n$, $n = 1, 100$ the indicator $I(u_n)$ was calculated, where

$$I\ (u_n) = \frac{Q_w^{-1}(u_n) - Q_m^{-1}(u_n)}{Q_w^{-1}(u_n)}\ 100\ , \qquad n = 1, \ldots, 100. \tag{34}$$

Note that the inverses of the indicators are used in the above expression so as to make their values consistent with indicators used by Joe and Liu [6] in that the values of the indicators go to zero as the tetrahedron degenerates:

$$I_{avg} = 0.01\Sigma I(u_n)\ ,\quad I_{max} = \ \max\ I(u_n), \quad n = 1, \ldots, 100. \tag{35}$$

Table 2 shows that the values of the two indicators differ by less than 10 percent but on occasion this difference may rise to 25 percent.

5.3. Anisotropic tetrahedra. In order to consider the case when the edge derivatives are nonuniform, consider the model tetrahedron defined by the four points

$$x_1 = [0, 0, 0]^T\ ,\ x_2 = [0, u, 0]^T,\ x_3 = \left[\frac{u}{2}, \frac{\sqrt{3}u}{2}, 0\right]^T\ ,\ \text{and}\ \ x_4 = \left[\frac{u}{2}, \frac{u}{2\sqrt{3}}, h\right]^T$$

$$\text{and}\quad h_1 = h_2 = h_3 = u \ \text{ and }\ h_4 = h_5 = h_6 = H = \sqrt{\frac{u^2}{3} + h^2}.$$

The volume of this tetrahedron is given by V, where $V = \frac{u^2 h}{4\sqrt{3}}$.

The anisotropy of the solution is represented by the assumption that the directional derivatives along the base differ greatly from those along the other edges, i.e.,

$$d_i = d_a u^2, \quad i = 1, 2, 3 \ \text{ and } \ d_i = d_b \ H^2, \quad i = 4, 5, 6,$$

and where it will be assumed that $|d_a u^2| >> |d_b H^2|$. Given these definitions, the anisotropy indicator has the value

$$Q_{aniso} = \left[\frac{12[\tilde{d}_a^2 u^4 + \frac{5}{4}\tilde{d}_a \tilde{d}_b \ H^2 u^2 + \tilde{d}_b^2 H^4]}{39 \ \max \ (d_a u^2, d_b H^2)^2} \right]. \tag{36}$$

Define the parameter $r = d_b H^2 / d_a u^2$ and rewrite the above as

$$Q_{aniso} = \frac{12}{39}\left[1 + 5/4 \ r + r^2\right]. \tag{37}$$

Hence, if r is small, then the anisotropy indicator identifies that a number of edges make a disproportionate contribution to the error.

6. Conclusions. The mesh quality indicators developed here appear to be a promising start in terms of identifying triangular or tetrahedral elements in which the shape of the elements and the local solution gradients conspire to give a poor linear approximation to a quadratic solution. The indicators have an obvious application in the case when linear triangular or tetrahedral finite elements are used to solve PDEs with anisotropic solutions.

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