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SOLUTION-BASED MESH QUALITY INDICATORS FOR TRIANGULAR AND TETRAHEDRAL MESHES

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ABSTRACT

A new mesh quality measure for triangular and tetrahedral meshes is presented. This mesh quality measure is based both on geometrical and solution information and is derived by considering the error when linear triangular and tetrahedral elements are used to approximate a quadratic function. The new measure is shown to be related to existing measures of mesh quality but with the advantage that local solution information in the form of scaled derivatives along edges is taken into account. This advantage is demonstrated by a comparison with a geometrical indicator on a parameterized problem.

Keywords: Unstructured meshes, tetrahedral mesh quality

1. Introduction

The increasing use of p.d.e. solvers based on triangular and tetrahedral meshes e.g see[4,17] raises the important issue of whether the mesh is appropriate to represent the solution. One approach to resolving this issue is have computable error estimates for each solution component. At present, it is still often the case that such estimates may not be available or may not be reliable. In the case of mesh generation, the usual approach is to assume that the solution to the problem is such that mesh quality may be viewed as being independent of the solution[5,11]. Indeed when no solution has been computed on the mesh this is the only way to proceed. Once a solution has been computed, the generally accepted point of view is that it is both the shape of the elements and the local solution behavior that is important, particularly for highly directional flow problems[14,15,16]. The starting point for this work was the analysis of Babuska and Aziz[3], who showed that the requirement for triangles was that there should be no large angles . This work was extended to tetrahedral elements by Krizek[10] in a similar spirit.

The intention here is not to deal with the issue of how to construct an optimal mesh but instead to consider the related issue of how an existing mesh should be assessed given a solution. This reflects an important practical issue, particularly in three space dimensions, when a mesh generator produces a mesh of unknown quality for a complex solution. The requirement is then to assess how appropriate the mesh is for the computed solution. The ideal solution is to use a computed error estimate to assess whether or not the mesh should be refined. This error estimate should reflect not only the interpolation error caused by approximating the solution by a finite element space on a given mesh but also the discretization error of the numerical method used to approximate the p.d.e. and the choice of norm used to measure the error.

In many cases however such error estimates are not available but it is still desirable to understand whether or not the mesh is appropriate. This paper will discuss the simple mesh quality indicator of Berzins[6] based on interpolation error estimates. The fundamental assumption being made is that the solution is being represented by a piecewise linear triangular or tetrahedral basis and that the function being approximated is quadratic. This assumption allows the error to be approximated by a quadratic function and the results of Nadler[12,13] to be used for the triangular case. The resulting indicator has been shown to be related to those of Bank[3] and Weatherill[17] when geometry alone is taken into account.

The quantities used in defining the full indicator have also been used to generate[8] and modify[2] meshes in two dimensions. This paper will show that the new indicator may be used to identify which triangular or tetrahedral element needs refining and also which edge(s) should be refined. A model of boundary layer flow will be used to demonstrate how the indicator performs in identifying which triangle is best. A further simple example will show the optimum mesh will depend critically on the choice of norm used to measure the error.

The second part of the paper will consider the indicator in the case of a linear element tetrahedral mesh. This indicator will again be shown to behave in a similar way to that of Weatherill[17]. A parameterized tetrahedron combined with a simple model of a solution with highly directional gradients will be used to illustrate how the new indicator identifies the effect of directionality on the linear element approximation error and how this contrasts with a purely geometrical mesh quality measure.

The conclusion of the paper is that while purely geometrical mesh quality indicators may do a good job in identifying meshing anomalies, the appropriateness of a mesh for a given solution cannot be decided using geometry alone.

2. Nadler's Error Estimate for Triangles.

The starting point for the derivation of a new mesh quality indicator is the work of Nadler[12] who derives a particularly appropriate expression for the interpolation error when a quadratic function is approximated by a piecewise linear function on a triangle. Consider the triangle T defined by the vertices v_1, v_2 and v_3 as shown in Figure 1 below. Let h_i be the length of the edge connecting v_i and v_{i+1} where $v_4 = v_1$.

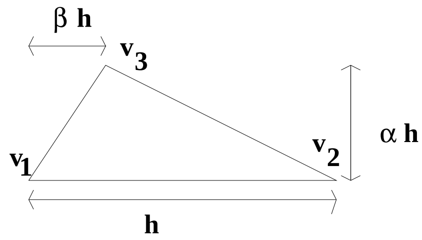

Figure 1: Babuska and Aziz Example Triangles.

Nadler[12] considers the case in which a quadratic function

$$u(x,y) \;=\; \frac{1}{2}\underline{x}^T \; H \; \underline{x} \; where \quad \underline{x} \;=\; \begin{bmatrix} x \\ y \end{bmatrix} \tag{1}$$

is approximated by a linear function $u_{lin}(x,y)$, as defined by linear interpolation based on the values of u at the vertices. Denote the error by

$$e_{lin}(x,y) = u_{lin}(x,y) \;-\; u(x,y) \tag{2}$$

Nadler[12] as quoted in Rippa[16] shows that

$$\int_T (e_{lin}(x,y))^2 dx \; dy \;=\; \frac{A}{180}\left[((d_1+d_2+d_3)^2 + {d_1}^2 + d_2^2 + d_3^3\right] \tag{3}$$

where A is the area of the triangle and $d_i = \frac{1}{2}(v_{i+1}-v_i)^T \; H \; (v_{i+1}-v_i)$ is the derivative along the edge connecting v_i and v_{i+1}.

Example 1 In the case when the matrix H is positive definite with diagonal entries p^2 and q^2 and symmetric off-diagonal entries pq then

$$d_i = (p \; \Delta x_i \;+\; q \; \Delta y_i)^2 \quad where \; v_{i+1} - v_i = [\Delta x_i, \Delta y_i]^T$$

In the case of the triangle in Figure 1 assuming that x and y are in the horizontal and vertical directions respectively, the values of d_i are $d_1 = p^2h^2$, $d_2 = h^2(-(1-\beta)p \; + \alpha q)^2$ and $d_3 = h^2(\beta p - \alpha q)^2$.

Example 2 In contrast when the matrix H has diagonal entries p and p and symmetric off-diagonal entries q then the matrix H has eigenvalues $p+q$ and $p-q$ and so is positive definite if $p > q$. In the case of the triangle in Figure 1 assuming that x and y are in the horizontal and vertical directions respectively, the values of d_i for this matrix are

$$d_1 = ph_1^2 \; , \; d_2 = \alpha^2 h_1^2(p(1+\mu_1^2) - 2\mu_1 q) \; and \; d_3 = \alpha^2 h_1^2(p(1+\mu_2^2) - 2\mu_2 q) \; . \tag{4}$$

where

$$\mu_1 = (1-\beta)/\alpha, \; and \quad \mu_2 \;=\; \beta/\alpha$$

In this case d_2 and d_3 can be negative if both p and q are positive and $q \gg p$. It is also possible to pick α and μ_1 so that $d_1 + d_2 + d_3 = 0$ in this case and hence zeros part of equation (3).

3. A Mesh Quality Indicator for Linear Triangular Elements

In this section the new mesh quality indicator of Berzins[6] based on the work of Nadler[12] will be derived. This indicator takes into account both the geometry and the solution behavior. The starting point for this indicator is equation (4): in the case when the values of d_i are all equal then each edge makes an equal contribution to the error. However in order to take into account in a consistent way the fact that the values of d_i may be of different signs it is necessary to consider their absolute values. It should also be noticed that if $d_i = h_i$ then the form of equation (see3) has some similarities with the indicators of Bank[3] and Weatherill[17]. This relationship will be made precise below. With these two points in mind the scaled forms of the derivatives d_i are defined by

$$\tilde{d}_i = \frac{|d_i|}{d_{max}} \; where \; d_{max} \; = \; max\,[|d_1|, |d_2|, |d_3|] \tag{5}$$

For notational convenience define

$$\tilde{q}(\underline{\tilde{d}}) \; = \; (\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3)^2 + \tilde{d}_1^2 + \tilde{d}_2^2 + \tilde{d}_3^2 \tag{6}$$

where $\underline{\tilde{d}} = [\tilde{d}_1, \tilde{d}_2, \tilde{d}_3]^T$. A measure of the anisotropy in the derivative contributions to the error is then provided by

$$q_{aniso} \; = \; \frac{\tilde{q}(\underline{\tilde{d}})}{12} \tag{7}$$

The definitions of the coefficients $\tilde{d}_i$ in equation (5) results in the bounds

$$\frac{1}{6} \; \leq \; q_{aniso} \; \leq \; 1 \tag{8}$$

Consider a triangle with only one edge contributing to the error. In this case $q_{aniso} = 1/6$ whereas if two edges contribute equally and the third makes no contribution $q_{aniso} = 1/2$.

In order to derive a consistent and related but geometry-only based indicator it should be observed that the quantity defined by:

$$q_m(\underline{h}) = \frac{\tilde{q}(\underline{h})}{16\,\sqrt{3}\,A}$$

where $\underline{h} = [h_1, h_2, h_3]^T$, has value 1 for an equilateral triangle and tends to the value infinity as the area of a triangle tends to zero but at least one of its sides is constant. It is now possible to explain the relationship between this indicator and those of Bank[3] and Weatherill[17] as denoted by q_b and q_w and defined by

$$\frac{1}{q_b} \; = \; \frac{1}{4\,\sqrt{3}\,A}\left[h_1^2 + h_2^2 + h_3^2\right], \qquad q_w \; = \; \frac{1}{3\,A}\left[(h_1 + h_2 + h_3)^2\right] \tag{9}$$

respectively. Hence,

$$q_m(\underline{h}) = \; \frac{1}{4\,q_b} \; + \; q_w\,\frac{\sqrt{3}}{16} \tag{10}$$

The relationship between q_{aniso} and the linear interpolation error is that when the matrix H is positive definite, i.e. $d_i > 0$, then

$$q_{aniso} = \frac{15}{A\ d_{max}^2} \int_T (e_{lin}(x,y))^2 dx\ dy, \tag{11}$$

thus showing that the indicator is a scaled form of the interpolation error in this special case.

3.1. Edge Indices and Mesh Generation/Movement.

In the case when q_{aniso} is small then it is possible to define an edge index which indicates how much each edge contributes to the error. Suppose that in equation (6) all the values of the terms $\tilde{d}_i$ are identical, say, $\tilde{d}_{avg}$ then

$$\tilde{q}(\tilde{d}) \;=\; 12(\tilde{d}_{avg})^2 \tag{12}$$

Hence

$$\tilde{d}_{avg} \;=\; \sqrt{q_{aniso}} \tag{13}$$

The edge index for each edge is then denoted by $e_{ind}(i)$ and defined by

$$e_{ind}(i) \;=\; \frac{\tilde{d}_i}{\tilde{d}_{avg}}, \quad i = 1, 2, 3. \tag{14}$$

these edge indices thus indicate which edges should be refined to reduce the error. One recent method to take advantage of such local gradients is the modified Delaunay approach of Borouchaki et al.[8] in which the local gradient information, of the form of d_i values, is used in conjunction with the Delaunay mesh generator to compute highly stretched grids for anisotropic flows in two space dimensions. The results presented by Borouchaki et al. show that this approach can give good results on problems with highly directional flows.

It is possible to compare the approach adopted here with the recent mesh movement method of Ait-Ali-Yahia et al.[2] in which the H matrix is modified to be positive definite and edge indicators, defined in the notation used here by $d_i/\sqrt{\Delta x_i^2 + \Delta y_i^2}$, are used to move the mesh. This approach thus scales the edge error component by the edge length. Ait-Ali-Yahia et al.[2] interpret d_i as the edge length in the H norm. The scaling defined by equation (13), in contrast, scales $|d_i|$ by an averaging factor taken over all the edges in the triangle. In the case when H is not positive definite as in Example 2 of Section 2 if the original values of d_2 and d_3 are negative (i.e. $q > p$) then the effect of the approach of Ait-Ali-Yahia et al.[2] is to transpose q and p in the H matrix and hence in the definitions of d_1, d_2 and d_3 thus giving different values from those in Section 2:

$$d_1 = qh_1^2\ ,\ d_2 = \alpha^2 h_1^2 (q(1+\mu_1^2) - 2\mu_1 p)$$

$$and \quad d_3 = \alpha^2 h_1^2 (q(1+\mu_2^2) - 2\mu_2 p)\ .$$

where μ_1 and μ_2 are defined as in Example 2 of Section 2.

3.2. Boundary Layer Flow Example

The performance of this indicator may be illustrated by considering anisotropic flow, such as that in a viscous boundary layer, in which the three triangles defined as Case(a), Case(b) and Case(c) in Figure 2 are used to model a flow with a weak horizontal component $u_{xx} = 1$ an intermediate cross derivative $u_{xy} = 100$ and and a strong vertical component $u_{yy} = 10000$. Case(a) is representative of a triangle thought to be especially suitable for such flows while Case(b) is closer to the type of triangles produced by unstructured mesh generators. Table 1 shows the values of q_{aniso} for the three triangles as the height of the triangles α is varied. Also shown is the ratio of the L_2 errors for Case (a) and Case (b) divided by the error in Case(c). The table shows that in the case when $\alpha < 0.04$ triangles such as that in Case(b) are best in terms of interpolation error. and that when $\alpha > 0.04$ triangles such as that in Case(c) are best in terms of interpolation error. The mesh anisotropy indicator q_{aniso} values show how the error is distributed and that smaller values of this indicator seem preferable since then one or more edge derivatives are orthogonal to the strongly directional error. For very small values of α anisotropy is not a key factor as the dominant flow direction is then the horizontal one and not the vertical one.

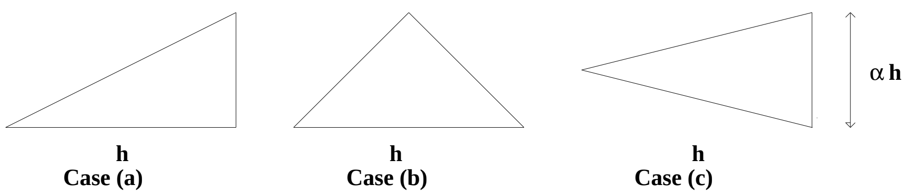

Figure 2: Boundary Layer Flow Example Triangles.

Table 1: Mesh Anisotropy Indicator Values

α	Case (a)	Case (b)	Case (c)	Error Ratio a/c	Error Ratio b/c
1.0	0.49	0.49	0.29	1.8	1.70
0.1	0.42	0.42	0.35	1.8	1.40
0.038	0.35	0.34	0.53	1.7	1.00
0.02	0.30	0.29	1.00	1.5	0.71
0.01	0.28	0.30	0.68	1.3	0.44
0.001	0.42	0.29	0.50	1.0	0.47
0.0001	0.49	0.28	0.50	1.0	0.55

3.3. Choice of Norm

The following example will illustrate how the choice of norm may be critical in deciding what is the best mesh by considering the H^1 and L_2 norms. Given the linear interpolation error defined by equation (2), The H^1 error norm is defined by

$||e_{lin}(x,y)||_{H^1}$ where

$$||e_{lin}(x,y)||^2_{H^1} = \int_T (e_{lin}(x,y))^2 + (e_{lin,x}(x,y))^2 + (e_{lin,y}(x,y))^2 dxdy \qquad (15)$$

The example used is that of Babuska and Aziz[3] in which triangles of the form of that in Figure 1 are used to model a flow with a horizontal component $u_{xx} = 1$ and no other non-zero components $u_{xy} = 0$ and $u_{yy} = 0$. In the notation of Babuska and Aziz $H = \alpha h$ in Figure 1 and the cases $\beta = 1$ and $\beta = \frac{1}{2}$ are considered. Hence $U(x,y) = \frac{1}{2}x^2$ and $U_{lin}(x,y) = \frac{1}{2}x + \beta(\beta-1)y/(2\alpha)$ and so

$$(e_{lin,x}(x,y))^2 \ + \ (e_{lin,y}(x,y))^2 \ = \ (x - \frac{1}{2})^2 \ + \ \beta(\beta-1)/(2\alpha)^2$$

thus showing a potential source of problems for small values of α. Berzins[7] shows that

$$||e_{lin}(x,y)||^2_{H_1} \ = \ \frac{A}{12}\left[\ \frac{1}{15}\tilde{q}(\underline{d}) \ + \ \tilde{r}(\underline{d}) \ \right] \qquad (16)$$

where for this problem

$$\tilde{q}(\underline{d}) = h^4\left[(1+\beta^2+(1-\beta)^2)^2 + (1-\beta)^4 + 1 + \beta^4\right] \qquad (17)$$

$$\tilde{r}(\underline{d}) = 4h^2\left[\frac{3\beta^2(1-\beta)^2}{\alpha^2}) + (1-\beta)^2 + \beta^2\right] \qquad (18)$$

and the term $\tilde{q}(\underline{d})$ is defined in equation (8). These results are interesting because they show that in the L_2 norm $\beta = \frac{1}{2}$ is more accurate whereas in the H^1 norm for $\alpha < 0.4629$, $\beta = 1$ or $\beta = 0$ is more accurate and $\beta = 0.5$ is the worst value as $\alpha \downarrow 0$. Hence a good mesh in one norm is not a good mesh in another norm.

3.4. Extensions to Non-Quadratic Functions

The extension to te case of non-quadratic functions may be considered by assuming that the exact solution is locally quadratic. Bank[4] uses such an approach inside the code PLTMG and calculates estimates of second derivatives. Adjerid, Babuska and Flaherty[1] use a similar approach based on derivative jumps across edges to estimate the error. An alternative is approach is to use the ideas of Hlavacek et al.[9] to estimate nodal derivatives and hence second derivatives.

4. Linear Tetrahedral Approximation of a Quadratic Function

The extension of Nadler's[12] approach to tetrahedra is achieved by considering the case in which a quadratic function

$$u(x,y,z) \ = \ \frac{1}{2}\ \underline{x}^T\ H\ \underline{x}\ where \quad \underline{x} \ = \ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (19)$$

is approximated by a linear function $u_{lin}(x,y,z)$ defined by linear interpolation based on the values of u at the vertices of a tetrahedron T defined by the vertices v_1, v_2 , v_3 and v_4 as shown in Figure 3.

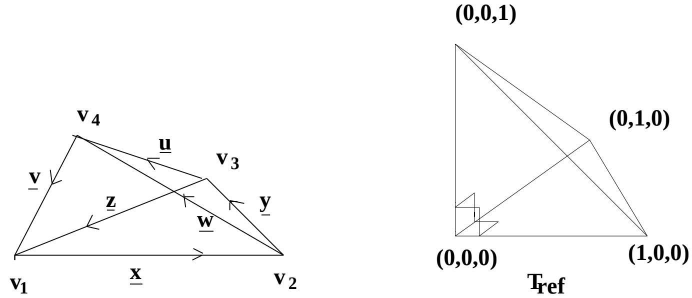

Figure 3: Example Tetrahedron and Reference Tetrahedron.

Let h_i be the length of the edge connecting v_i and v_{i+1} where $v_5 = v_1$. With reference to Figure 3 define the vectors $\underline{\hat{x}}$, $\underline{\hat{y}}$, $\underline{\hat{z}}$, $\underline{\hat{u}}$, $\underline{\hat{v}}$ and $\underline{\hat{w}}$ by

$$v_2 = v_1 + \underline{\hat{x}}, \quad v_3 = v_2 + \underline{\hat{y}}, \quad v_1 = v_3 + \underline{\hat{z}}$$

$$v_4 = v_1 - \underline{\hat{v}}, \quad v_4 = v_2 + \underline{\hat{w}}, \quad v_4 = v_3 + \underline{\hat{u}} \tag{20}$$

and consequently

$$\underline{\hat{x}} + \underline{\hat{y}} + \underline{\hat{z}} \;=\; \underline{\hat{x}} + \underline{\hat{w}} + \underline{\hat{v}} \;=\; \underline{\hat{u}} + \underline{\hat{v}} - \underline{\hat{z}} \;=\; 0. \tag{21}$$

Define a reference tetrahedron T_{ref}, see Figure 3, by the four nodal points:

$$v_1 = (0,0,0)^T \quad , \quad v_2 = (1,0,0)^T \quad , \quad v_3 = (0,1,0)^T \quad , \quad v_4 = (0,0,1)^T \tag{22}$$

Then the mapping from the tetrahedron, T_{ref}, to the tetrahedron, T is given by

$$\underline{x} = v_1 \;+\; B\,\underline{\tilde{x}} \tag{23}$$

where $B \;=\; [\underline{\hat{x}}, -\underline{\hat{z}}, -\underline{\hat{v}}]$, $\underline{\tilde{x}}$ is in the reference tetrahedron. T_{ref} and $\underline{x}$ is the equivalent point in the original tetrahedron T .
The function u may then be expressed as

$$u(x,y,z) \;=\; \frac{1}{2}\,v_1^T\;H\;v_1 \;+\; \frac{1}{2}\,\underline{\tilde{x}}^T\;B^T\;H\;v_1 \;+\; \frac{1}{2}\,v_1^T\;H\;B\,\underline{\tilde{x}} \;+\; \frac{1}{2}\,\underline{\tilde{x}}^T\;B^T\;H\;B\,\underline{\tilde{x}} \tag{24}$$

where $\underline{\tilde{x}} \;=\; [x,y,z]^T$, is defined on T_{ref}. Ignoring the constant and linear terms (which are approximated exactly by a linear interpolant and expanding the remaining quadratic term using equation (23) gives

$$\begin{aligned} u(x,y,z) \;=\; \frac{1}{2}\,&[(\underline{\hat{x}}^T H \underline{\hat{x}})x^2 + (-\underline{\hat{x}}^T H \underline{\hat{z}})2xy + (\underline{\hat{z}}^T H \underline{\hat{z}})y^2 \\ &(-\underline{\hat{x}}^T H \underline{\hat{v}})2xz + (\underline{\hat{z}}^T H \underline{\hat{v}})2zy + (\underline{\hat{v}}^T H \underline{\hat{v}})z^2 \;\;] \end{aligned}$$

Interpolating this by a linear function defined on T_{ref} by the nodal solution values gives

$$u_{lin}(x,y,z) \;=\; \frac{1}{2}\,\left[(\underline{\hat{x}}^T H \underline{\hat{x}})x + (\underline{\hat{z}}^T H \underline{\hat{z}})y + (\underline{\hat{v}}^T H \underline{\hat{v}})z\right] \tag{25}$$

and hence the linear interpolation error may be defined as as:

$$e_{lin}(x,y,z) = u_{lin}(x,y,z) \;-\; u(x,y,z) \tag{26}$$

and written as

$$\begin{aligned} e_{lin}(x,y,z) = \; & \frac{1}{2}\,[(\underline{\hat{x}}^T H \underline{\hat{x}})(x-x^2) \;-\; (-\underline{\hat{x}}^T H \underline{\hat{z}})2xy \;+\; (\underline{\hat{z}}^T H \underline{\hat{z}})(y-y^2) \\ & -\; (-\underline{\hat{x}}^T H \underline{\hat{v}})2xz \;-\; (\underline{\hat{z}}^T H \underline{\hat{v}})2zy \;+\; (\underline{\hat{v}}^T H \underline{\hat{v}})(z-z^2)\;\;] \end{aligned}$$

This in turn may be written as

$$e_{lin}(x,y,z) = \;\frac{1}{2}\,\underline{W}^T\;\underline{\hat{d}} \tag{27}$$

where

$$\underline{W}^T = \left[x-x^2,\;\; -2xy,\;\; y-y^2,\;\; -2xz,\;\; -2zy,\;\; z-z^2\right]$$

and

$$\underline{\hat{d}}^T \;\;= \left[\underline{\hat{x}}^T H \underline{\hat{x}}, -\underline{\hat{x}}^T H \underline{\hat{z}}, \underline{\hat{z}}^T H \underline{\hat{z}}, -\underline{\hat{x}}^T H \underline{\hat{v}}, \underline{\hat{z}}^T H \underline{\hat{v}}, \underline{\hat{v}}^T H \underline{\hat{v}}\;\right]$$

Hence from equation (27)

$$\int_T \;(e_{lin}(x,y,z))^2 dx\;dy\;dz = \;\frac{6V}{4}\;\int_{T_{ref}} \underline{\hat{d}}^T\;\underline{W}\;\underline{W}^T\;\underline{\hat{d}}\;\;dx\;dy\;dz \tag{28}$$

where V is the volume of the tetrahedron. Berzins[6] shows that this may then be written as

$$\int_T \;(e_{lin}(x,y))^2 dx\;dy\;dz = \;\frac{3}{2}\;V\;\underline{d}^T\;\;P\;\underline{d}\;\;\;. \tag{29}$$

where the vector of second directional derivatives along edges is defined by

$$\underline{d}^T = \frac{1}{2}\;[d_1,...,d_6\;] \;\;=\;\; \frac{1}{2}\;\left[\underline{\hat{x}}^T H \underline{\hat{x}},\;\;\underline{\hat{y}}^T H \underline{\hat{y}},\;\;\underline{\hat{z}}^T H \underline{\hat{z}},\;\;\underline{\hat{u}}^T H \underline{\hat{u}},\;\;\underline{\hat{v}}^T H \underline{\hat{v}},\;\;\underline{\hat{w}}^T H \underline{\hat{w}}\;\right].$$

and where the matrix P is defined by

$$P \;\;=\;\; \frac{4}{7!}\;\begin{bmatrix} 4 & 2 & 2 & 1 & 2 & 2 \\ 2 & 4 & 2 & 2 & 1 & 2 \\ 2 & 2 & 4 & 2 & 2 & 1 \\ 1 & 2 & 2 & 4 & 2 & 2 \\ 2 & 1 & 2 & 2 & 4 & 2 \\ 2 & 2 & 1 & 2 & 2 & 4 \end{bmatrix}$$

Expanding out equation (29) in terms of the components of $\underline{d}$ which are the six directional derivatives along the edges gives:

$$\int_T \;(e_{lin}(x,y,z))^2 dx\;dy\;dz = \;\frac{6}{4}\;V\;\frac{8}{7!}\;\left[(\Sigma d_i)^2 - d_1 d_4 - d_2 d_5 - d_3 d_6 + \Sigma d_i^2\;\right]\;\;\;. \tag{30}$$

5. Tetrahedral Mesh Quality Indicator

The results in the previous section make it possible to define the mesh quality indicator in the same way as in Section 2 in that the error is scaled by the maximum directional derivative d_{max}, the integral is scaled by the volume before taking the square root. In a similar way to that found in Section 3 define

$$\tilde{Q}(\underline{\tilde{d}}) \;=\; \left[(\Sigma\tilde{d}_i)^2 - \tilde{d}_1\tilde{d}_4 - \tilde{d}_2\tilde{d}_5 - \tilde{d}_3\tilde{d}_6 + \Sigma\tilde{d}_i^{\;2}\right] \tag{31}$$

where now $\underline{\tilde{d}} = [\tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \tilde{d}_4, \tilde{d}_5, \tilde{d}_6]^T$. A measure of the anisotropy in the derivative contributions to the error is then provided by

$$Q_{aniso} \;=\; \frac{\tilde{Q}(\underline{\tilde{d}})}{39} \tag{32}$$

By defining the normalized derivatives as in equation (3) a geometry based indicator can be written as

$$Q_m(\underline{h}) = \frac{C}{V}\left[\tilde{Q}(\underline{\tilde{h}})\right]^{\frac{3}{2}} \tag{33}$$

where C is a scaling factor to ensure that the indicator has value one when $h_i = h$ and thus $C = 1/(8.48528 \times 39^{1.5})$ and the power of $\frac{3}{2}$ reflects the different dimensions of the error and the volume in powers of h.

The edge quality estimator used by Weatherill[17] is of the form

$$Q_w = \frac{1}{8.48528V}\left[(\Sigma\frac{h_i}{6})^3\right] \;. \tag{34}$$

A comparison of these two indicators on tetrahedra with uniform gradients was done by Berzins[6] using the eight parameterized tetrahedra of Liu and Joe[11] as defined by Figures 4 to 11 of that paper and showing that the values of the two indicators differ by less than ten percent but on rare occasions that this difference may rise to 25 percent.

5.1. Edge Indices

As in two dimensions it is possible to define an edge index which indicates how much each edge contributes to the error. Suppose that in equation (31) all the values of $\tilde{d}_i$ are identical, say, $\tilde{d}_{avg}$ then

$$\tilde{Q}(\underline{\tilde{d}}) = 39(\tilde{d}_{avg})^2 \tag{35}$$

Hence

$$\tilde{d}_{avg} \;=\; \sqrt{Q_{aniso}} \;. \tag{36}$$

The edge index for each edge is then denoted by $e_{ind}(i)$ and defined by

$$e_{ind}(i) \;=\; \frac{\tilde{d}_i}{\tilde{d}_{avg}}, \quad i = 1, 2, 3, 4, 5, 6. \tag{37}$$

5.2. Anisotropic Tetrahedra

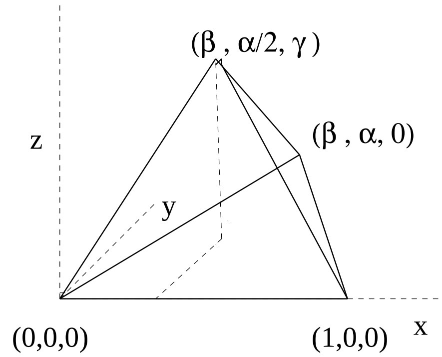

Figure 4: Example Anisotropic Tetrahedron

In order to consider the case when the edge derivatives are nonuniform consider the model tetrahedron in Figure 4 defined by the four points

$$x_1 = [0,0,0]^T \ , \ x_2 = [1,0,0]^T, \ x_3 = [\beta, \alpha, 0]^T \ and \ \ x_4 = [\beta, \frac{\alpha}{2}, \gamma]^T$$

with edge lengths

$$h_1 = 1, h_2 = \sqrt{\alpha^2 + (1-\beta)^2}, \quad h_3 = \sqrt{\alpha^2 + \beta^2},$$

$$h_4 = \sqrt{\alpha^2/4 + \gamma^2}, \ h_5 = \sqrt{\alpha^2/4 + \beta^2 + \gamma^2}, \ and \ h_6 = \sqrt{\alpha^2/4 + (1-\beta)^2 + \gamma^2},$$

The volume of this tetrahedron is given by V where $V \ = \ \alpha\gamma/6$. The anisotropy of the solution is shown by the fact that the directional derivatives d_i given below depend only on β and not on γ or α.

$$d_1 = 1/2, d_2 = 1/2(1-\beta)^2, d_3 = 1/2\beta^2,$$

$$d_4 = 0, \ d_4 = 1/2\beta^2, \ and \ d_6 = 1/2(1-\beta)^2$$

Given these definitions the anisotropy indicator has the value shown in the table below. In contrast a geometry based indicator such as that of Weatherill, will for small values of α and β indicate a possible source of problems, as is shown in Table 2 below. Table 2 also shows the values of the H^1 norm which is defined as in Section 3.3 except that there is now a third gradient term $(e_{lin,z}(x,y))^2$ and the gradient terms sum to

$$\int_T (e_{lin,x}(x,y))^2 + (e_{lin,y}(x,y))^2 + (e_{lin,z}(x,y))^2 dxdy \ = \ \frac{V}{4} \times$$

$$\left[0.05 + 1.2(\beta - 0.5)^2 + \beta^2(1-\beta)^2(\frac{1}{\alpha^2} + \frac{1}{\gamma^2}) \right] \tag{38}$$

Hence as in Section 3.3 this norm is sensitive to small values of α and/or γ. Yet again the behavior of the error norms exhibits different trends from the indicator Q_w. Thus again suggesting that the error norm be used to identify which elements should be refined and the anisotropy indicator and the values of d_i to determine which edges should be targetted for refinement.

Table 2: Q_{aniso} , Standard Mesh Quality Q_w and Square of L_2 and H^1 Norm Values

		$\beta = 0$			$\beta = 0.5$		
Indicator	α/γ	0.01	1.0	100.0	0.01	1.0	100.0
Q_{aniso}		0.28	0.28	0.28	0.12	0.12	0.12
	0.01	9.1e+2	5.2e+1	9.0e+4	8.9e+2	4.7e+1	9.0e+4
Q_w	1.00	5.5e+1	1.20	9.1e+2	4.5e+1	1.03	9.1e+2
o	100.	1.4e+5	1.4e+3	5.0e+1	1.4e+5	1.4e+3	5.1e+1
Scaled	0.01	1.0e-4	1.0e-2	1.0	1.0e-4	1.0e-2	1.0
L_2^2	1.00	1.0e-2	1.0	1.0e+2	1.0e-2	1.0	1.0e+2
error	100.	1.0	1.0e+2	1.0e+4	1.0	1.0e+2	1.0e+4
Scaled	0.01	1.0e-4	1.0e-2	1.0	6.9e-1	3.5e+1	3.5e+3
$(H^1)^2$	1.00	1.0e-2	1.0	1.0e+2	3.5e+1	1.0	6.5e+1
error	100.	1.0	1.0e+2	1.0e+4	3.5e+3	6.5e+1	3.0e+3

In Table 2 the L_2 and H^1 norms for each value of $beta = 0.0, 0.5$ are scaled by the value of the norm when $\alpha = \gamma = 1$. This makes a comparison with the mesh quality Q_w indicator easier as it has a values close to 1 at these points.

The indicators Q_w , Q_{aniso} and the L_2 error are symmetric about $\beta = 0.5$. In particular when α and γ are small then $h_i \approx d_i$ and

$$Q_w = \frac{1}{8.48528V} \left[(\Sigma \frac{d_i}{6})^3 \right] \quad . \tag{39}$$

Hence as the volume shrinks the mesh quality indicator Q_w becomes large while the approximation error for a fixed value of β is scaled only by the volume. The most significant result is that the indicator Q_{aniso} doesn't vary with α and γ and the error norm naturally increases as α, γ and hence the volume get large. In contrast the mesh quality indicator Q_w has a minimum when $\alpha = \gamma = 1$ and is also relatively small when α and γ are large and the error is also large.

6. Conclusions

The mesh quality indicators developed here appear to be a promising start in terms of identifying triangular or tetrahedral elements in which the shape of the elements and the local solution gradients conspire to give a poor linear approximation to a quadratic solution. The indicators have an obvious application in the case when linear triangular or tetrahedral finite elements are used to solve p.d.e.s with anisotropic solutions. The differences that occur when the indicator is used in the very simplest cases suggest that it is important to try to include solution effects when assessing the quality of the mesh.

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