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The unknown wavefunction of a single system cannot be inferred using a series of quantum measurements

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Recently, Aharonov, Anandan and Vaidman [1,2] showed that the wavefunction of a single quantum system could be determined from the results of a series of "protective measurements" performed on the system. In the protective measurement scheme, a-priori knowledge of the wavefunction of the system is used in order to measure this system and protect its wavefunction from changing at the same time. Aharonov, Anandan and Vaidman argued that the protective measurement accounts for the physical reality of the wavefunction. Yet, it seems that one should be able to measure the wavefunction of a single system without any a-priori knowledge, if the wavefunction were real.

According to the projection postulate, a precise measurement of a single system would always yield one of the eigenvalues of the measured observable. This eigenvalue can be used to estimate the expectation value of the observable. The uncertainty of the measured observable could never be estimated using the single measurement result. This uncertainty cannot be estimated even if we use the results of additional measurements of the single system. After the measurement, the wavefunction of the system collapses to the eigenstate, which corresponds to the measured eigenvalue, and the results of additional measurements would not add any information about the initial wavefunction of the system. Now, consider a very weak measurement of the single system. This measurement would leave the wavefunction of the system almost unchanged. Consecutive weak measurements of the same system, therefore, would give us some additional information about the initial wavefunction of the system. Since the measured system is approximately in the same state when the different measurements are performed, one may expect the statistics of the measurement results to be approximately the same as the results of an ensemble measurement, where a single measurement is performed on each system in an ensemble of identical systems. Specifically, one may expect the statistics of these measurement results to enable us to estimate the uncertainties of the measured observables with finite estimate errors, where the ability to estimate the uncertainty of a specific observable distinguishes a measurement of the wavefunction from a measurement of the observable.

In this work, we show that this intuitive picture fails, and one cannot, in fact, determine the initial wavefunction of a single system at all using a series of weak quantum measurements. We prove that the statistics of the results of a series of measurements of a single system are independent of the initial uncertainties of the measured observables, as a direct result of the projection postulate.

Consider a series of alternating measurements of the two conjugate observables $\hat{q}$ and $\hat{p}$ of a single quantum system, which is initially in the pure state described by the density operator $\hat{\rho}_0$. The statistics of the $\hat{q}$ measurement results are expected to give information about the initial probability density of $\hat{q}$, $P_0(q) = {}_s\langle q|\hat{\rho}_0|q\rangle_s$, i.e., estimates of the initial center position of this probability density (or the expectation value of $\hat{q}$), $\langle q_0\rangle = \int dq\, P_0(q)\, q$, and the initial width of the probability density (or the uncertainty of $\hat{q}$), $\langle \Delta q_0^2\rangle = \langle q_0^2\rangle - \langle q_0\rangle^2$, where $\langle q_0^2\rangle = \int dq\, P_0(q)\, q^2$. In the same way, the statistics of the $\hat{p}$ measurement results are expected to give information about $P_0(p) = {}_s\langle p|\hat{\rho}_0|p\rangle_s$. Note that this model applies to the case of a series of measurements of the observable $\hat{q}\cos\theta + \hat{p}\sin\theta$, for all $\theta \in [0, 2\pi]$. Indeed, one needs, at least, information about the probability densities of all of these observables in order to reconstruct the wavefunction.

First, $\hat{q}$ is measured. The measured system is correlated to a probe, and after the correlation the probe is measured to yield the inferred measurement result $\tilde{q}_1$. The probability-amplitude operator, $\hat{Y} = {}_p\langle \tilde{q}_1|\hat{U}|\phi\rangle_p$, completely describes the three stages of this measurement [3]: The preparation of the probe in the state $|\phi\rangle_p$, the interaction of the probe with the measured system, $\hat{U}$, and the result of the measurement, $\tilde{q}_1$, which corresponds to the state of the probe after the measurement, $|\tilde{q}_1\rangle_p$. The probability of obtaining the measurement result $\tilde{q}_1$ is

$$P(\tilde{q}_1) = \mathrm{Tr}_s[\hat{Y}\hat{\rho}_0\hat{Y}^\dagger] \ . \tag{1}$$

Note that the statistics of the results of an ensemble measurement would be $P(\tilde{q}_1)$. In general, though, the measurement process disturbs the wavefunction of the measured system, and it is not necessarily possible to infer $P_0(q)$ from $P(\tilde{q}_1)$.

Let us, therefore, consider first the case in which the measurements are back-action evading measurements [4], i.e., $[\hat{U}, \hat{q}] = 0$, and

$$P(\tilde{q}_1) = \int dq\, X(q, \tilde{q}_1)\, P_0(q) \ , \tag{2}$$

where $X(q, \tilde{q}_1) = {}_s\langle q|\hat{Y}^\dagger\hat{Y}|q\rangle_s$ is the probability for the probe to undergo a transition from $|\phi\rangle_p$ to $|\tilde{q}_1\rangle_p$ when the signal is in the state $|q\rangle_s$. In this case one can use $P(\tilde{q}_1)$ to infer $\langle q_0\rangle$ and $\langle \Delta q_0^2\rangle$, the initial expectation value and uncertainty of $\hat{q}$. We also assume that the measurements satisfy the following three conditions. First, the transition probability of the probe is required to be normalized over all possible final states of the probe,

$$\int d\tilde{q}_1\, X(q, \tilde{q}_1) = 1 \ . \tag{3}$$

As the inferred value of $\hat{q}$, $\tilde{q}_1$ should equal, on average, the center position of the probability density of $\hat{q}$, $\langle \tilde{q}_1\rangle = \int d\tilde{q}_1\, P(\tilde{q}_1)\, \tilde{q}_1 = \langle q_0\rangle$. This leads to the second condition,

$$\int d\tilde{q}_1\, X(q, \tilde{q}_1)\, \tilde{q}_1 = q \ . \tag{4}$$

The system and the probe should be independent of each other. Therefore, the probability error associated with the measurement result, $\tilde{q}_1$, should equal the sum of the measurement error, Δ_m^2, and the intrinsic uncertainty of $\hat{q}$, $\langle \Delta\tilde{q}_1^2\rangle = \langle \tilde{q}_1^2\rangle - \langle \tilde{q}_1\rangle^2 = \langle \Delta q_0^2\rangle + \Delta_m^2$, where $\langle \tilde{q}_1^2\rangle = \int d\tilde{q}_1\, P(\tilde{q}_1)\, \tilde{q}_1^2$. From this we obtain the third condition,

$$\int d\tilde{q}_1\, X(q, \tilde{q}_1)\, \tilde{q}_1^2 = q^2 + \Delta_m^2 \ . \tag{5}$$

After the first measurement, the system is described by the density operator

$$\hat{\rho} = P(\tilde{q}_1)^{-1}\, \hat{Y} \hat{\rho}_0 \hat{Y}^\dagger \ . \tag{6}$$

The corresponding probability density of $\hat{q}$ is $P(q,\tilde{q}_1) = {}_s\langle q|\hat{\rho}|q\rangle_s = P(\tilde{q}_1)^{-1}\, X(q,\tilde{q}_1)\, P_0(q)$. Note that $P(q,\tilde{q}_1)$ depends on $\tilde{q}_1$. The measurement process, therefore, modifies the wave-function of the measured system according to the measurement result. Next, the conjugate observable $\hat{p}$ is measured. We assume that the change in the probability density of $\hat{q}$ from $P(q,\tilde{q}_1)$ to $P_1(q,\tilde{q}_1)$ is the minimum change possible. In this case, the center position is unchanged, $\int dq\, P_1(q,\tilde{q}_1)\, q = \int dq\, P(q,\tilde{q}_1)\, q$, but the width increases due to Δ_b^2, the back-action noise, $\int dq\, P_1(q,\tilde{q}_1)\, q^2 = \int dq\, P(q,\tilde{q}_1)\, q^2 + \Delta_b^2$. Now $\hat{q}$ is measured for the second time. Following the treatment of the first measurement of $\hat{q}$ in Eqs. (2)-(5), the conditional probability to obtain $\tilde{q}_2$ in this measurement, after $\tilde{q}_1$ is obtained in the previous measurement, is

$$P(\tilde{q}_2|\tilde{q}_1) = \int dq\, X(q,\tilde{q}_2)\, P_1(q,\tilde{q}_1) \ . \tag{7}$$

Now, consider the statistics of $\tilde{q}_1$ and $\tilde{q}_2$. Obviously, each of the measurement results, $\tilde{q}_1$ or $\tilde{q}_2$, estimates the initial center position, $\langle q_0\rangle$, since $\langle\tilde{q}_1\rangle = \langle\tilde{q}_2\rangle = \langle q_0\rangle$. Also, one can estimate the second order moment $\langle q_0^2\rangle$ using either $\tilde{q}_1$ or $\tilde{q}_2$, since

$$\langle\tilde{q}_1^2\rangle = \int d\tilde{q}_1\, P(\tilde{q}_1)\, \tilde{q}_1^2 = \langle q_0^2\rangle + \Delta_m^2 \ , \tag{8}$$

$$\langle\tilde{q}_2^2\rangle = \int d\tilde{q}_1\, P(\tilde{q}_1) \int d\tilde{q}_2\, P(\tilde{q}_2|\tilde{q}_1)\, \tilde{q}_2^2 = \langle q_0^2\rangle + \Delta_m^2 + \Delta_b^2 \ . \tag{9}$$

However, one cannot estimate the initial width, $\langle\Delta q_0^2\rangle$, using a single measurement result, because a single measurement result does not contain information about $\langle q_0\rangle^2$. If $\tilde{q}_1$ and $\tilde{q}_2$ were independent results, obtained from two different quantum systems, which are initially in the same quantum state, their correlation would provide the missing information about $\langle q_0\rangle^2$, and $\langle\Delta q_0^2\rangle$ could be estimated using both measurement results. In our case the second measurement result, $\tilde{q}_2$, depends on the first, $\tilde{q}_1$, and their correlation does not give information about $\langle q_0\rangle^2$, rather it gives

$$\langle\tilde{q}_1\tilde{q}_2\rangle = \int d\tilde{q}_1\, P(\tilde{q}_1)\, \tilde{q}_1 \int d\tilde{q}_2\, P(\tilde{q}_2|\tilde{q}_1)\, \tilde{q}_2 = \langle q_0^2\rangle \ . \tag{10}$$

In fact, in order for $\tilde{q}_1\tilde{q}_2$ to give an estimate of $\langle q_0\rangle^2$, the transition probability $X(q,\tilde{q})$ should be a function of $\langle q_0\rangle$. This is impossible, since $X(q,\tilde{q})$ does not depend on the state of the measured system, as can be seen from Eq. (2). The conditions of Eqs. (3)-(5) are, therefore, not necessary for our conclusion to be valid. This treatment can be easily extended to include as many measurements of $\hat{q}$ as we want, by way of mathematical induction, and a similar treatment can be used to analyze the results of the measurements of $\hat{p}$. Always the conclusion is the same: While it is possible to estimate the initial center positions of $P_0(q)$ and $P_0(p)$ with a linear function of the corresponding measurement results, no quadratic function of the measurement results can estimate the initial widths of $P_0(q)$ and $P_0(p)$. Since no information about the widths of the probability densities is obtained, the process of repeated measurements is equivalent to measurements of the observables $\hat{q}$ and $\hat{p}$, and cannot be considered as a determination of the wavefunction.

Let us consider now the case in which the back-action evading condition, $[\hat{U}, \hat{q}] = 0$, does not necessarily hold.

Using Eq. (1), the probability to obtain $\tilde{q}_1$ in the first measurement can be written in general as $P(\tilde{q}_1) \equiv \int dq\, \delta(q - \tilde{q}_1)P(q) = \int dq\, {}_s\langle q|\hat{Y}\hat{\rho}_0\hat{Y}^\dagger|q\rangle_s$. The probability density of $\hat{q}$ after this measurement is $P_1(q) = P(\tilde{q}_1)^{-1}\, \delta(q - \tilde{q}_1)\, P(q)$, where Eq. (6) is used. The next measurement is a precise measurement of $\hat{q}$ which results with $\tilde{q}_2$. $P(\tilde{q}_2|\tilde{q}_1) = \int dq\, \delta(q - \tilde{q}_2)\, P_1(q)$ is the conditional probability to obtain $\tilde{q}_2$ in this measurement. The statistics of $\tilde{q}_1$ and $\tilde{q}_2$ give $\langle \tilde{q}_1 \rangle = \langle \tilde{q}_2 \rangle$ and $\langle \tilde{q}_1^2 \rangle = \langle \tilde{q}_2^2 \rangle = \langle \tilde{q}_1 \tilde{q}_2 \rangle$. These statistics show that with or without the second measurement result, the variance of $P(\tilde{q}_1)$ cannot be estimated due to the lack of an estimate of $\langle \tilde{q}_1 \rangle^2$, since $\tilde{q}_2$ depends on $\tilde{q}_1$. While an ensemble measurement gives the probability density $P(\tilde{q}_1)$ from which $P_0(q)$, the initial probability density of the measured observable, may be inferred, a series of measurements of a single system does not, since the wavefunction of the measured system changes each time a measurement is performed in accordance with the measurement result. This change cannot be corrected for using unitary time evolution in between the two measurements without a-priori knowledge of the initial wavefunction of the measured system, just as one cannot devise a protective measurement [1,2] for a system in an unknown state. It may be possible to correct for that change using a measurement process, as in the case of the "reversible measurements" that was suggested by Royer [5]. However, the probability that the measurement process be reversed successfully is finite, and it is not certain that the wavefunction of the measured system will return to its original unknown state. Taking into account this finite success probability Huttner [6] showed, that the statistics of the results of a series of successfully reversed measurements of a single system are independent of the initial state of the measured system, and cannot be used to infer the initial wavefunction of the system.

To conclude, we have shown that the unknown wavefunction of a single system cannot be inferred using a series of quantum measurements. Each time a measurement is performed, the wavefunction changes in accordance with the measurement result. Therefore, the statistics of the measurement results contain no information about the initial uncertainties of the measured observables. This physical mechanism, which originates in the projection postulate, limits the quantum wavefunction to have a statistical meaning only.

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