ACHILLES AND THE TURTLE IN THE MAZE OF THE WEAK MEASUREMENT

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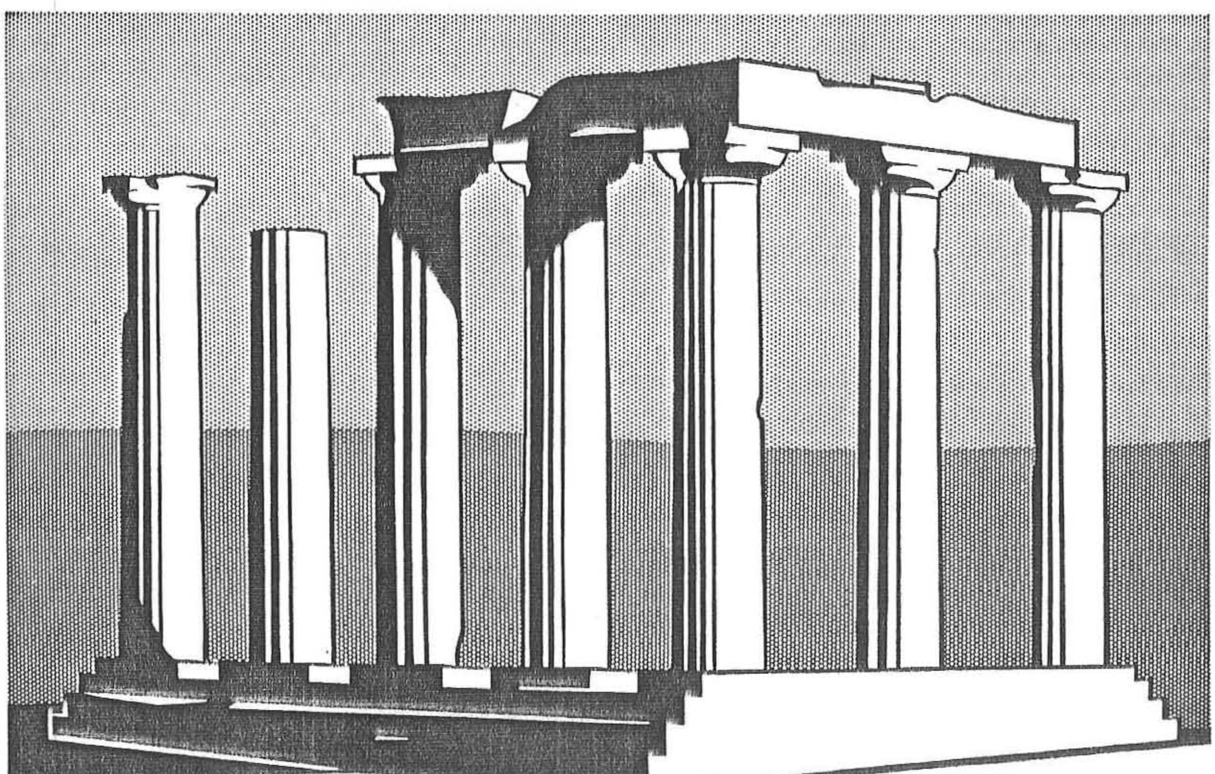
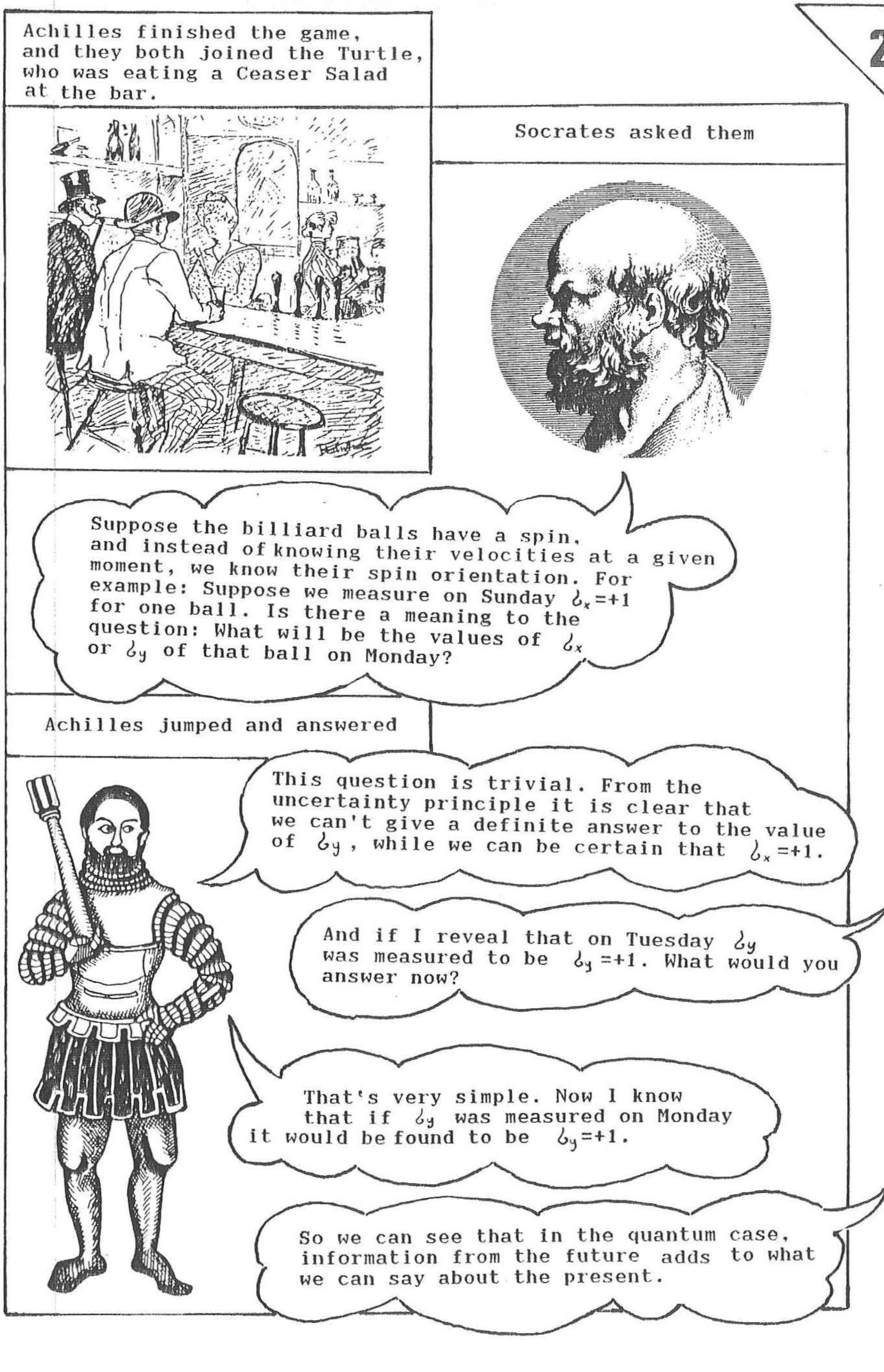
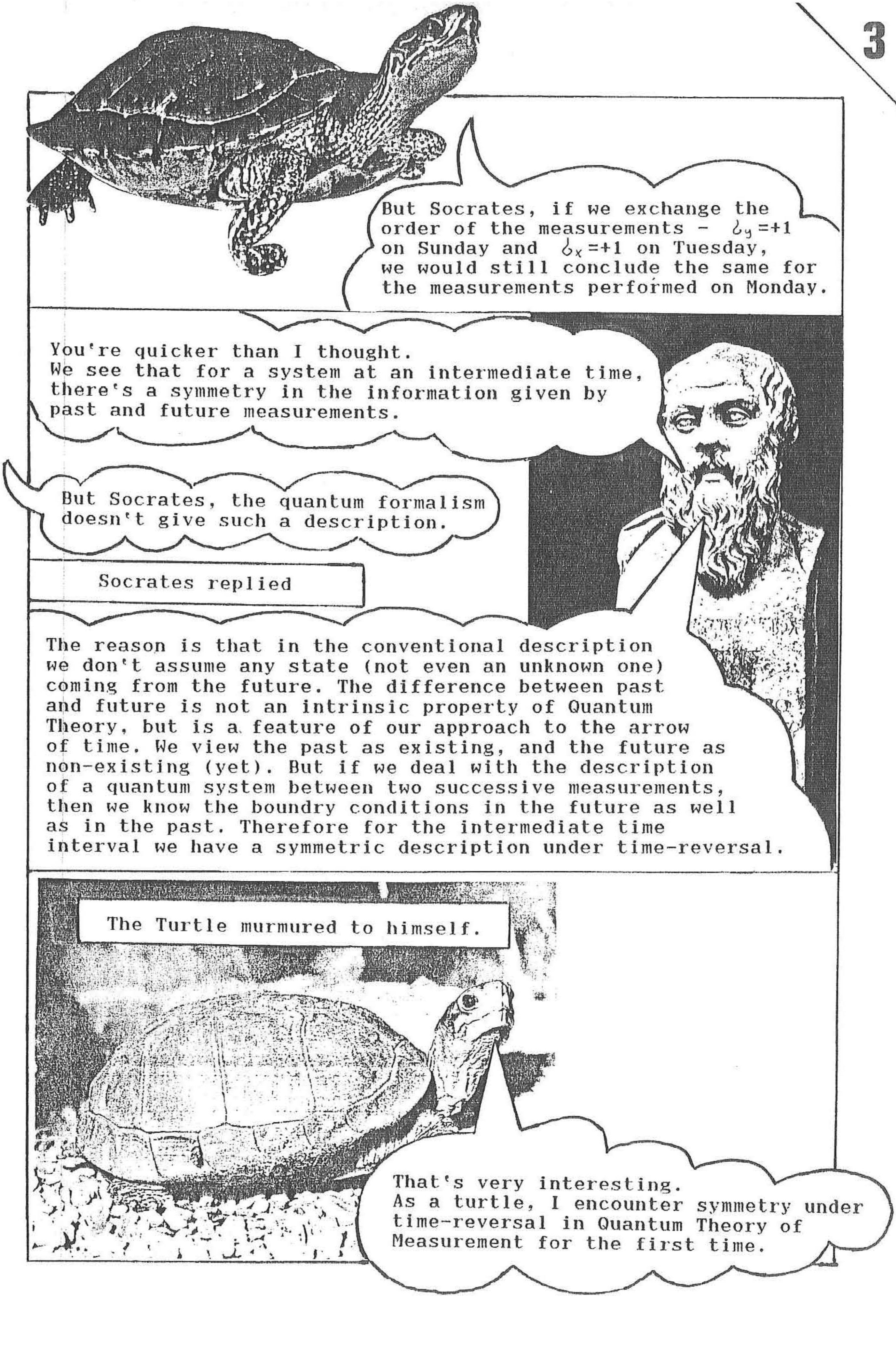



One sunny afternoon Socrates wandered into the bar, when Achilles was about to win a pool game.
Socrates greeted him
I see that you're lucky today!
Lucky? It's time you learned something from others, Socrates. I'm the master of calculated shots. I know how to hit this red ball, so that it will hit the black one and roll it right into the hole.
I understand that from the moment you hit the red ball, there's no meaning to the question whether the black ball went into the hole. All you need in order to know the state of the balls in any given moment, is their state at the moment you hit one of them.
That's what makes me the champion, Socrates.


Achilles finished the game, and they both joined the Turtle, who was eating a Ceaser Salad at the bar.
Socrates asked them
Suppose the billiard balls have a spin, and instead of knowing their velocities at a given moment, we know their spin orientation. For example: Suppose we measure on Sunday σ_x=+1 for one ball. Is there a meaning to the question: What will be the values of σ_x or σ_y of that ball on Monday?
Achilles jumped and answered
This question is trivial. From the uncertainty principle it is clear that we can't give a definite answer to the value of σ_y, while we can be certain that σ_x=+1.
And if I reveal that on Tuesday σ_y was measured to be σ_y=+1. What would you answer now?
That's very simple. Now I know that if σ_y was measured on Monday it would be found to be σ_y=+1.
So we can see that in the quantum case, information from the future adds to what we can say about the present.


But Socrates, if we exchange the order of the measurements - σ_y=+1 on Sunday and σ_x=+1 on Tuesday, we would still conclude the same for the measurements performed on Monday.
You're quicker than I thought. We see that for a system at an intermediate time, there's a symmetry in the information given by past and future measurements.
But Socrates, the quantum formalism doesn't give such a description.
Socrates replied
The reason is that in the conventional description we don't assume any state (not even an unknown one) coming from the future. The difference between past and future is not an intrinsic property of Quantum Theory, but is a feature of our approach to the arrow of time. We view the past as existing, and the future as non-existing (yet). But if we deal with the description of a quantum system between two successive measurements, then we know the boundry conditions in the future as well as in the past. Therefore for the intermediate time interval we have a symmetric description under time-reversal.
The Turtle murmured to himself.
That's very interesting. As a turtle, I encounter symmetry under time-reversal in Quantum Theory of Measurement for the first time.

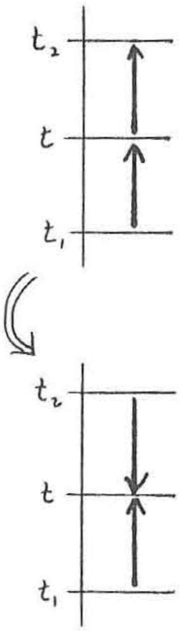
Socrates continued

Let's describe three consecutive measurements in the conventional formalism:

a. First measurement at t_1, after which the system is at the state $|\psi_1\rangle$.

b. An intermediate measurement at t, of the observable A – $A|\alpha_i\rangle = \alpha_i|\alpha_i\rangle$.

c. Second measurement at t_2, after which the system is at the state $|\psi_2\rangle$.



The probability for the transition from the state $|\psi_1\rangle$ to the state $|\psi_2\rangle$ via the state $|\alpha_i\rangle$ is:

$$P_i = |\langle\psi_2|U(t_2,t)|\alpha_i\rangle\langle\alpha_i|U(t,t_1)|\psi_1\rangle|^2$$

Note that it can also be written in the form:

$$P_i = |\langle U^\dagger(t_2,t)\psi_2|\alpha_i\rangle\langle\alpha_i|U(t,t_1)\psi_1\rangle|^2$$

This form expresses an important change in our measurement's description: Here we describe $|\psi_2\rangle$ evolving backwards in time towards the intermediate measurement (Which also implies that the collapse of the wave-function due to a measurement evolves in both directions of time symmetrically).
If we repeat this serie of measurements many times we'll get an ensemble, for which the specific results of the boundry measurements at t_1 (preselection) and at t_2 (postselection), define the sub-ensemble that interests us. Normalization of P_i in this sub-ensemble will be defined by -

$$P_i' = \frac{P_i}{\sum_k P_k}$$

Achilles became enthusiastic

With this new description we can really see that the measurments at times t_1 and t_2 form the boundry conditions of the system at any intermediate time. It makes the symmetry so evident.

That's really nice,
but we haven't seen any new results that emerge from this new description.

Your complaint is a serious one. But I claim that the new formalism will enable us to predict interesting results which would have been difficult to anticipate using the conventional formalism. The new formalism will also give us a better intuitive understanding of these results.

Socrates continued

For this purpose let us define a new type of measurement: The Weak Measurement. Weak, in the sense that it does not significantly disturb the state of the system. If before the measurement the system was at the state $|\Psi\rangle$, then after the measurement it will be at a state only slightly different from $|\Psi\rangle$.
The first new result is for the weak measurements of non-commuting variables: $[A,B]\neq 0$.
To a system where a measurement of A at time t_1 gave $A(t_1)=\alpha$, and a measurement of B at $t_2>t_1$ gave $B(t_2)=\beta$, let us add two intermrdiate weak measurements: $B_\omega(t)$ and afterwards $A_\omega(t')$, where $t_2>t'>t>t_1$. Since B_ω is a weak measurement, it won't change the state $|\alpha\rangle$, and we can be certain that the result of $A_\omega(t')$ will be α . Symmetrically we can be certain of the result $B_\omega(t)=\beta$. Even if we exchange the order of the weak measurements ($A(t_1)$; $A_\omega(t)$; $B_\omega(t')$; $B(t_2)$), their results would still be the same. But the later would also be the case if all the measurements were strong. Surprisingly, we found that the order of weak measurements is not important even if we deal with non-commuting variables. This means that we could perform weak measurements of non-commuting variables simultaneously.

Isn't there a contradiction here with the uncertainty principle?

No. In our experiments the error in the result of a single weak measurement is greater than the minimal uncertainty due to the Heisenberg principle, so that $\Delta A \cdot \Delta B >> \hbar$. Weakening the interaction reduces the accuracy of a single measurement, so that it provides almost no information. In order to get meaningfull information from such measurements we will have to perform them on an ensemble of identical particles.

The Turtle doesn't seem to understand.

How do we know that a measurement is a weak one?

How would you define a measurement?

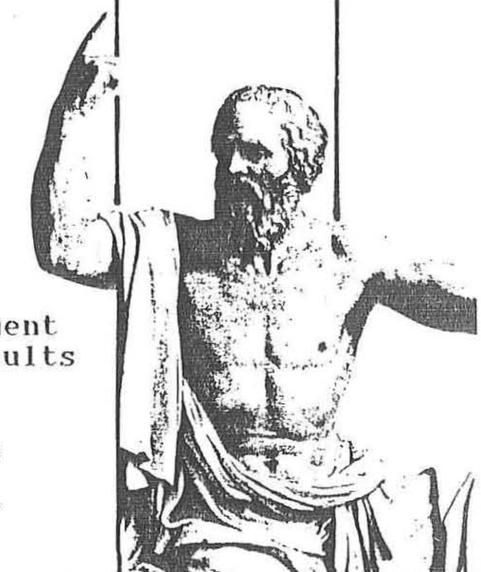
Achilles, you make me wonder whether you didn't take part in Prof. Aharonov's course "Quantum Measurement Theory." Let me remind you briefly: a measurement is an interaction of a system with a measuring device. In the case of a linear interaction the interaction Hamiltonian is: $H=-g(t)qA$, where A is the measured varible of the system, and q is the canonical variable of the measurement device which interacts with A. π, the cojugate momentum of q, will change in time according to Hamilton's equation:

$$\frac{d\pi}{dt} = -\frac{\partial H}{\partial q} = g(t)A$$

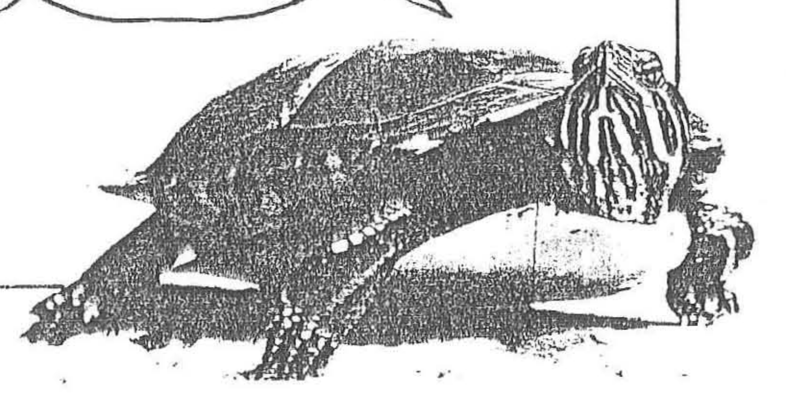
We read the result from the π scale of the measuring device, that is

$$A \propto \delta\pi \equiv \pi_f - \pi_i .$$

From here it is evident that a precise measurement of A means setting precise π values, which results in a large uncertainty in q (significant probability for large q values).
If we want a weak measurement, we can reason in the reverse order and demand a preparation of the measuring device so that the probability of measuring large q values is small enough.



From $H_{int}=-g(t)qA$ it is easy to realise that small q values mean weak coupling. For example in a Stern-Gerlach experiment $H_{int}=-g(t)\underline{B}\cdot\underline{\mu}$, and our demand for small q values represents a weak magnetic field.



Many functions have a small spread around zero. For the sake of simplicity we shall choose the initial state of the measuring device to be a Gaussian of q with a spread Δ.

$$|MD(t=0)\rangle \propto \exp\left[-\left(\frac{q}{2\Delta}\right)^2\right]$$

Now the initial total state of the measuring device and the measured system will be:

$$|\Psi_1\rangle |MD(t=0)\rangle .$$

The time-evolution operator for this state is:

$$U(T) = \exp\left(-i\int_0^T H dt\right)$$

Adding the post selection requirement, that the system is at the state $|\Psi_2\rangle$ at the end of the measurement, we will get the final state of the measuring device:

$$\langle\Psi_2|\exp\left(-i\int_0^T H dt\right)|\Psi_1\rangle \exp\left[-\left(\frac{q}{2\Delta}\right)^2\right] =$$

$$= \langle\Psi_2|\exp(iqA)|\Psi_1\rangle \exp\left[-\left(\frac{q}{2\Delta}\right)^2\right] =$$

$$= \langle\Psi_2|\Psi_1\rangle \exp(iqA_\omega)\exp\left[-\left(\frac{q}{2\Delta}\right)^2\right] + \langle\Psi_2|\Psi_1\rangle \left\{\sum_{n=2}^{\infty}\frac{(iq)^n}{n!}\left[(A^n)_\omega - (A_\omega)^n\right]\right\} \exp\left[-\left(\frac{q}{2\Delta}\right)^2\right]$$

when - $A_\omega \equiv \frac{\langle\Psi_2|A|\Psi_1\rangle}{\langle\Psi_2|\Psi_1\rangle}$

Achilles interupts him

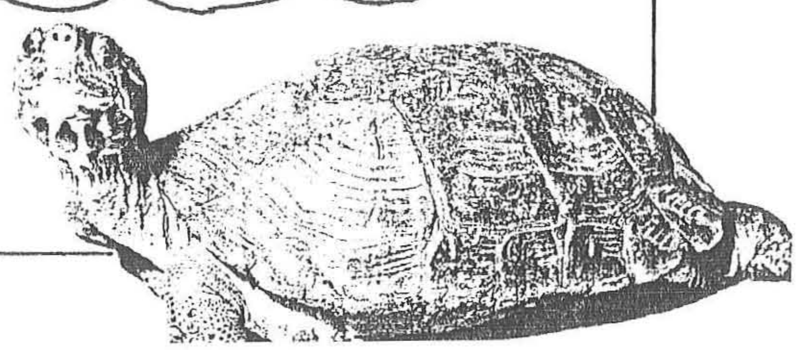
Wait a minute. What's A_ω?

As we can see from the second line of the mathematical derivation, the total effect of the measurement is expressed in exp(iqA). Notice that if the sum component ($\sum_{n=2}^{\infty}\ldots$) is neglected, then we will get that the total effect of the measurement is exp(iqA_ω). The condition for neglecting the sum component is:

$$(2\Delta)^n \frac{\Gamma(\frac{n}{2})}{(n-2)!}\left|(A^n)_\omega - (A_\omega)^n\right| << 1, \quad \forall n \geq 2$$

The Turtle is puzzled

Is this the condition for a measurement to be weak?



That is a good question. The weakness condition was initially defined by preparing the q state of the measuring device with a small spread Δ around zero. I remind you that this was in order to minimize the disturbance of the system's state by the measurement. To be precise, we want a neglectable variation of $|\psi_1\rangle$ (forward projected), $|\psi_2\rangle$ (backward projected) and $\langle\psi_2|\psi_1\rangle$, across the weak measurement. But what will be the result of such a measurement? We are looking for a universal result in order to give the weak measurement a meaning. The condition we imposed on Δ -

$$(2\Delta)^n \frac{\Gamma(\frac{n}{2})}{(n-2)!} \left|(A^n)_w - (A_w)^n\right| \ll 1, \quad \forall n \geq 2$$

gave us $A_w \equiv \frac{\langle\psi_2|A|\psi_1\rangle}{\langle\psi_2|\psi_1\rangle}$ as the universal result. As we shall see, this expression has interesting properties.

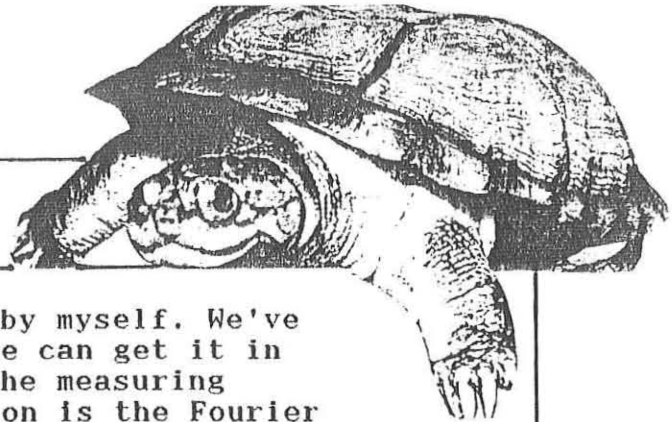
I notice that the derivation we made before is exact and that we've used no approximations.

Good. In fact we can see the result of any measurement as a sum of two contributions: the first expresses the interaction with the undisturbed state -

$$\langle\psi_2|\psi_1\rangle \exp(iqA_w) \exp\left[-\left(\frac{q}{2\Delta}\right)^2\right]$$

and the second expresses the interaction with all of the disturbed states -

$$\langle\psi_2|\psi_1\rangle \left\{ \sum_{n=2}^{\infty} \frac{(iq)^n}{n!} \left[(A^n)_w - (A_w)^n\right] \exp\left[-\left(\frac{q}{2\Delta}\right)^2\right]\right..$$



The Turtle is stoked

I can finish the derivation by myself. We've said $A \propto \delta\pi$. Let's see how we can get it in here: The initial state of the measuring device in its π representation is the Fourier transform of the q representation -

$$\exp\left[-\left(\frac{q}{2\Delta}\right)^2\right] \longrightarrow \exp\left(-\Delta^2 \pi^2\right)$$

The final state in its π representation is a Gaussian, whose center is moved by $Re(A_\omega)$ from the center of the initial Gaussian -

$$\exp\left(iq A_\omega\right) \exp\left[-\left(\frac{q}{2\Delta}\right)^2\right] \longrightarrow \exp\left[-\Delta^2 (\pi - A_\omega)^2\right].$$

Socrates is stoked too

I've promised you interesting features of $A\omega$...

Achilles interupts him

I think I can see one. The $A\omega$ value $\frac{\langle\Psi_2|A|\Psi_1\rangle}{\langle\Psi_2|\Psi_1\rangle}$, is not limited even when the A Spectrum is bounded. This is because of the product in the denominator which can be as small as we wish (according to the transition probability from the initial to the final state). For example, for spin measurement we can choose the initial state - $|\Psi_1\rangle = |\sigma_z = +1\rangle$
the final state - $|\Psi_2\rangle = |\sigma_{z\theta} = +1\rangle$
and perform a weak measurement of σ_θ .
The result will be

$$(\sigma_\theta)_\omega \approx \frac{1}{\cos\theta}$$

This means we can get the surprising result $(\sigma_\theta)_\omega \gg 1$!

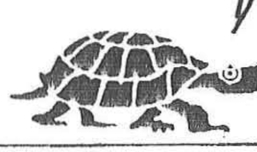
The Turtle turtlizes

Can we check these ideas experimentally? I can't see how it is possible to perform the postselection.

Actually there are two ways of realizing our procedure. Remember that we should perform it on an ensemble of systems in order to reduce the uncertainty in the measured value A_ω.
One possibility is to measure each system separately by coupling it to its own measuring device. With these measuring devices we shall perform the three measurements: preselection at t_1, postselection at t_2 and a weak measurement at time t ($t_1<t<t_2$). We shall consider the results of the weak measurements of only the systems in which the pre and post measurements gave the specific boundry states we have defined.
The second possibility is to perform one weak measurement on the whole ensemble. Here the weakness condition is for the interaction of each individual system with the measuring device. For example, the interaction of a compass' needle with each spin in a ferromagnet. In this case we can prepare the ferromagnet at t_1 with magnetization along the x-axis (preselection). We will get the postselection by considering only those cases in which the ferromagnet had its magnetization along the y-axis at t_2.

Actually the magnetization measured by the compass' needle is the average of the magnetic moments of all the individual systems (spins).

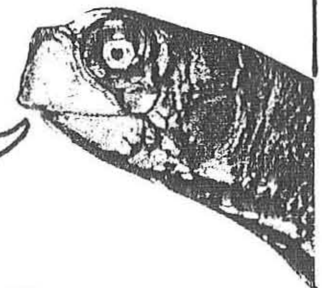
Obviously, almost always when you measure an ensemble of identical systems with a single measuring device, the result is an average value. For every operator A, we can define the average operator: $\frac{1}{N}\sum_{i=1}^{N} A_i$



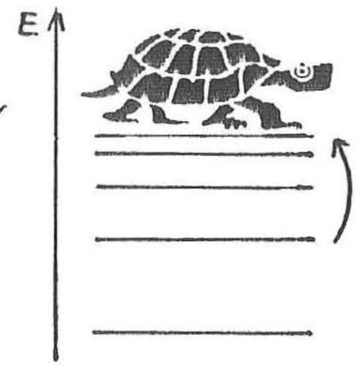
Very Good, Turtle. And what will be the interaction Hamiltonian in such a case?

Oh! Well...

$$H = -g(t)\, q\, \frac{1}{N} \sum_{i=1}^{N} A_i$$



And the Turtle gets excited



From this expression we can see that the interaction of the measuring device with each individual system is reduced by $\frac{1}{N}$. This means that measuring the average operator over a large ensemble is a weak measurement of each individual system.

Quite right, Turtle. We can show mathematically that since an operator's action can be written as:

$$A|\Psi\rangle = \bar{A}|\Psi\rangle + \Delta A\, |\Psi_\perp\rangle$$

and if the ensemble's state is $\prod_{i=1}^{N} |\Psi_i\rangle$ the action of the average operator on it will give:

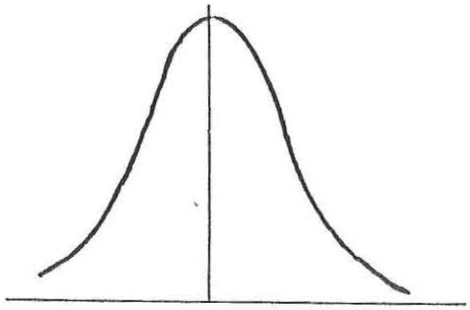
$$\frac{1}{N} \sum_{i=1}^{N} A_i \left(\prod_{i=1}^{N} |\Psi_i\rangle \right) = \bar{A} \left(\prod_{i=1}^{N} |\Psi_i\rangle \right) + \frac{\Delta A}{N} \left(|\Psi_{i\perp}\rangle \prod_{k \neq i} |\Psi_k\rangle \right)$$

For large N the second term can be neglected, and the state of the system after the measurement is similar to what it was before the measurement. As you recall, the requirement of not disturbing significantly the system's state is how we defined the weak measurement.

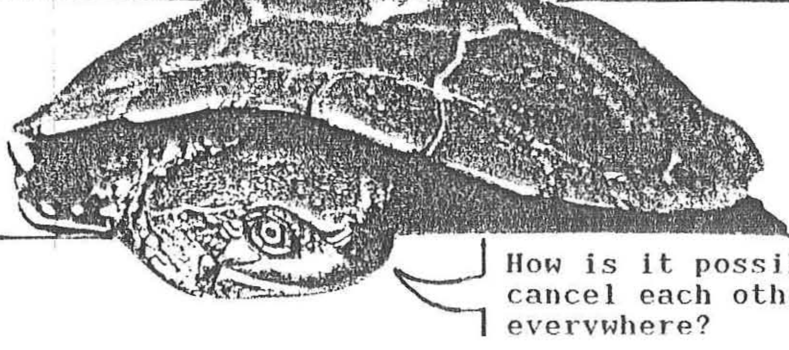
Achilles scratched his nose.

I'm still bothered by the fact that A_ω could be very large, even when the A Spectrum is bounded! Isn't there a contradiction with the conventional quantum formalism?

As you've seen, the mathematics we've used so far is the same mathematics of the conventional formalism. Therefore, there has to be a way to get this special result of A_ω using the conventional formalism. We have chosen the distribution of the quantum variable to be a Gaussian with a well defined center. Still the Gaussian has non-zero values even far from the center.

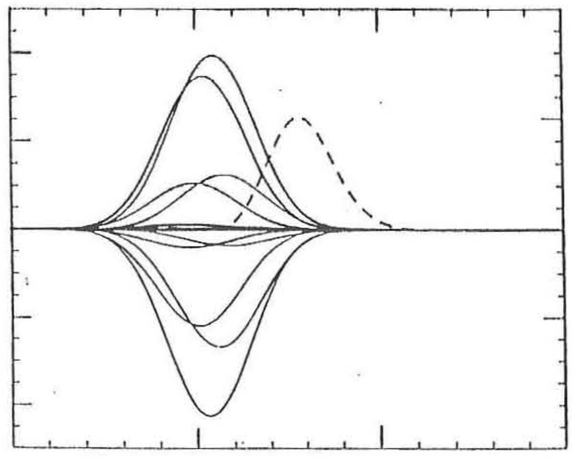


The spectrum of the average operators is discrete and bounded. For example, the average spin of an ensemble $\frac{1}{N}\sum_i \sigma_i$ can obtain values from -1 to 1, with equal steps of $\frac{2}{N}$.
In the conventional quantum description each discrete eigenvalue will shift the Gaussian's center respectively. It turns out that describing the weak measurement in the conventional formalism will result in such Gaussians, that in superposition will cancel everywhere except in a certain small region of the discrete values' domain.



How is it possible that Gaussians cancel each other if they are positive everywhere?

The measurement process is expressed by the factor exp $(-i\int_0^T H dt)$. Each different eigenvalue of the Hamiltonian, will give a different relative phase to its corresponding Gaussian. This is how, with the appropriate relative phases, Gaussians can cancel each other out. Again we see the importance of the relative phases of states in quantum theory and their responsibility for its unique phenomena.



A mathematical "miracle": superposition of Gaussians shifted by values between -1 and 1 equals to a Gaussian shifted by the value 3.

(Aharonov, Anandan, Vaidman [3]).

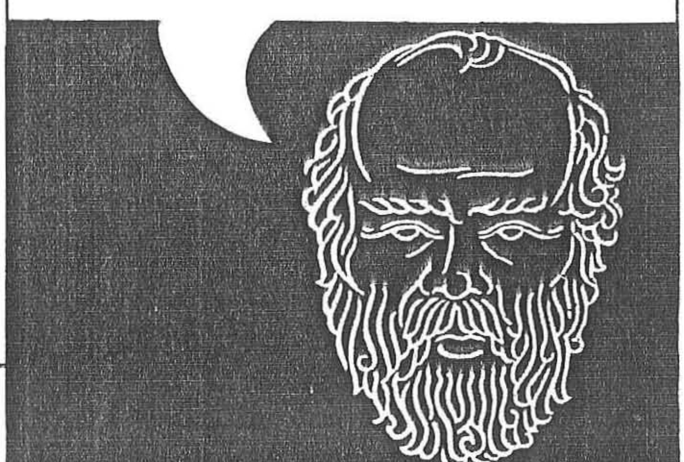
The waiter arrives with the bill.

Sorry, if we don't close now, we'll be fined.

Achilles yawns and the Turtle stretches in his chair.

Socrates smiles.

O.K. guys, I see that you are tired. But if you are interested you can check out the refrences.



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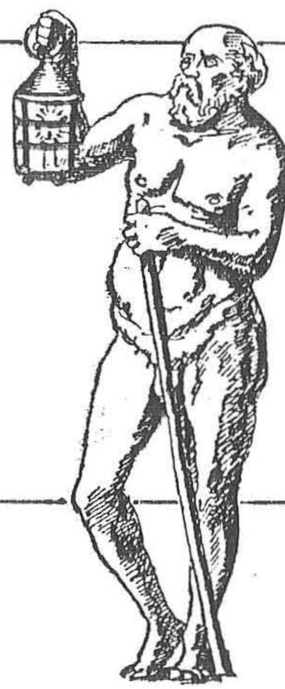

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