JOURNAL OF COMPLEXITY 18, 641–659 (2002)
doi:10.1006/jcom.2001.0625

A Two-Dimensional Bisection Envelope Algorithm for Fixed Points

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Received February 5, 2001; revised August 21, 2001; accepted August 21, 2001; published online March 13, 2002

In this paper we present a new algorithm for the two-dimensional fixed point problem $f(x)=x$ on the domain $[0,1]\times[0,1]$, where f is a Lipschitz continuous function with respect to the infinity norm, with constant 1. The computed approximation $\tilde{x}$ satisfies $\|f(\tilde{x})-\tilde{x}\|_\infty \leqslant \varepsilon$ for a specified tolerance $\varepsilon<0.5$. The upper bound on the number of required function evaluations is given by $2\lceil\log_2(1/\varepsilon)\rceil+1$. Similar bounds were derived for the case of the 2-norm by Z. Huang *et al.* (1999, *J. Complexity* **15**, 200–213), our bound is the first for the infinity norm case.

1. INTRODUCTION

The development of constructive algorithms for approximating fixed points started in the 1920s with Banach's simple iteration algorithm. Several algorithms have been developed since then, including homotopy continuation, simplicial and Newton type methods [1–3, 8]. It has been shown [4] that for Lipschitz functions with constant $q>1$ with respect to the infinity norm, the latter algorithms exhibit exponential complexity in the worst case (when computing ε-residual solutions $\tilde{x}$: $\|\tilde{x}-f(\tilde{x})\|_\infty \leqslant \varepsilon$) and that the lower bound on the complexity is also exponential.

Several algorithms for approximating a fixed point α of a Lipschitz function that is contractive ($q<1$) or nonexpanding ($q=1$) with respect to the second norm have been developed [5, 9, 10]. For $q<1$ and large dimension d, the Banach simple iteration algorithm $x_{i+1}=f(x_i)$ is optimal [7, 9]. It requires $n=\lceil\log(1/\varepsilon)/\log(1/q)\rceil$ function evaluations to compute $\tilde{x}$ such that $\|\tilde{x}-\alpha\|_2 \leqslant \varepsilon\,\|\alpha\|_2$. In the univariate case a class of bisection-envelope algorithms is optimal with respect to various error criteria [9]. For moderate dimension d and $q=1$, the interior ellipsoid algorithm is

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optimal [5, 6]. This algorithm requires $c \cdot d \cdot \log(1/\varepsilon)$ function evaluations to compute an ε-residual approximation $\tilde{x}$: $\|\tilde{x} - f(\tilde{x})\|_2 \leqslant \varepsilon$. We stress that the worst case complexity of computing an ε-absolute approximation x: $\|\tilde{x} - \alpha\|_2 \leqslant \varepsilon$ for $q = 1$ is infinite [9]. This means that there exists no algorithm based on function evaluations that solves this problem for all functions in this class. For $q < 1$ the interior ellipsoid algorithm [5] computes $\tilde{x}$: $\|\tilde{x} - \alpha\| \leqslant \varepsilon$ within $c \cdot d(\log(1/\varepsilon) + \log(1/(1-q)))$ function evaluations.

The above results hold for the case of the second norm. The case of the infinity norm and $q \leqslant 1$ requires further research. In this paper we present a two-dimensional bisection-envelope algorithm that exhibits worst case complexity $2\lceil \log_2(1/\varepsilon) \rceil + 1$ for the residual error criterion, $q = 1$ and the infinity norm. We believe that this bound is optimal, and conjecture that the optimal d-dimensional lower bound is $\mathcal{O}(c(d) \log(1/\varepsilon))$, where $c(d)$ is a polynomial in d with small degree. A subsequent conclusion is that the complexity with the absolute error criterion, $q < 1$, and small d is at most $\mathcal{O}(c(d)(\log(1/\varepsilon) + \log(1/(1-q))))$. We believe that the complexity for the absolute error criterion and $q = 1$ is infinite as in the case of the second norm. By following the proof for the second norm [9] we can conclude that this is indeed the case for $q > 1$.

2. PROBLEM FORMULATION

Given $a, b \in \mathbb{R}$ with $a < b$, we define the domain $D_{a,b} \equiv [a, b]^2$ and consider the class of Lipschitz continuous functions

$$\mathfrak{F}_{a,b} \equiv \{f \colon D_{a,b} \to D_{a,b} : \|f(x) - f(y)\| \leqslant \|x - y\| \ \forall x, y \in D_{a,b}\}, \tag{1}$$

where $\|\cdot\| = \|\cdot\|_\infty$ henceforth. We define $D \equiv D_{0,1}$ and $\mathfrak{F} \equiv \mathfrak{F}_{0,1}$. Any $f \in \mathfrak{F}_{a,b}$ maps $D_{a,b}$ into $D_{a,b}$, so by the Brouwer fixed point theorem there exists $x^* \in D_{a,b}$ such that $f(x^*) = x^*$.

In this paper we present an algorithm which, for every $f \in \mathfrak{F}$ and for a given positive $\varepsilon < 0.5$, computes a solution $\tilde{x} = \tilde{x}(f) \in D$ satisfying the residual criterion,

$$\|f(\tilde{x}) - \tilde{x}\| \leqslant \varepsilon. \tag{2}$$

Under certain conditions the computed solution also satisfies the absolute criterion,

$$\|\tilde{x} - \alpha\| \leqslant \varepsilon \tag{3}$$

for a fixed point α of f. (If $\varepsilon \geqslant 0.5$ then $\tilde{x} = (0.5, 0.5)$ clearly satisfies (2) and (3).) The algorithm requires n evaluations of f, where

$$1 \leqslant n \leqslant 2\lceil \log_2 (1/\varepsilon) \rceil + 1. \tag{4}$$

3. PRELIMINARY RESULTS

In this section we prove certain fixed point properties of the functions in $\mathfrak{F}_{a,b}$.

3.1. *Envelope Theorem*

For a given $f \in \mathfrak{F}_{a,b}$ we define the fixed point sets

$$F_1(f) \equiv \{x \in D_{a,b} : f_1(x) = x_1\},$$

$$F_2(f) \equiv \{x \in D_{a,b} : f_2(x) = x_2\},$$

$$F(f) \equiv F_1(f) \cap F_2(f).$$

We observe that $F(f)$ is the set of all fixed points of f, and that $F(f) \neq \varnothing$.

In Fig. 1, given $f \in \mathfrak{F}$, we illustrate envelopes of the graph of f_1 after it is evaluated at some point $x' \in D$. The variables in the graph are $z = (z_1, z_2, z_3) \in [0, 1]^3$. The plane L consists of the points $\{z \in [0, 1]^3 : z_1 = z_3\}$. We assume that $f_1(x') > x'_1$, so that the point $z' = (x'_1, x'_2, f_1(x'))$ lies above L. We define the set $P \equiv \{z \in [0, 1]^3 : z'_3 - z_3 \geqslant \|(z'_1, z'_2) - (z_1, z_2)\|\}$, which resembles a pyramid descending from z'. By Lipschitz continuity of f, Int(P) cannot contain a point z such that $z_3 = f_1(z_1, z_2)$. It follows that if $z \in \text{Int}(P) \cap L$ then $z_1 \neq f_1(z_1, z_2)$. We let Q be the projection onto the (x_1, x_2)-plane of the intersection of P and L. Then for all $x \in \text{Int}(Q)$ we have $f_1(x) \neq x_1$. Clearly $x' \in Q$. Let T be the triangular set $\{x: |x'_2 - x_2| \leqslant x'_1 - x_1\}$. Then $T \subset Q$. We formalize this in Theorem 3.1, the "envelope theorem."

THEOREM 3.1. *For any $f \in \mathfrak{F}_{a,b}$ and any permutation (i, j) of* (1,2)*, let $x \in D_{a,b}$ be such that $f_i(x) \neq x_i$. Then*

(i) *If $f_i(x) > x_i$ then for every $y \in D_{a,b}$ such that $\|y - x\| < (f_i(x) - x_i)/2$, the set*

$$A_i(y) \equiv \{z \in D_{a,b} : |y_j - z_j| \leqslant y_i - z_i\}$$

does not intersect $F_i(f)$. (Observe that Int(Q) $= \bigcup_y A_i(y)$.)

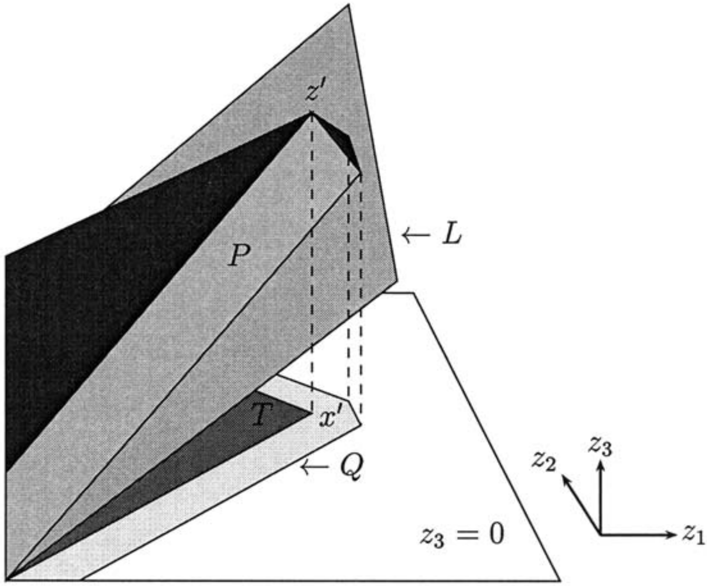


FIG. 1. Graphical representation of Theorem 3.1.

(ii) *If* $f_i(x) < x_i$ *then for every* $y \in D_{a,b}$ *such that* $\|y - x\| < (x_i - f_i(x))/2$, *the set*

$$B_i(y) \equiv \{z \in D_{a,b} : |z_j - y_j| \leqslant z_i - y_i\}$$

does not intersect $F_i(f)$. *(Observe that* $Int(Q) = \bigcup_y A_i(y)$.*)*

Proof. To show (i) we take y with $\|y - x\| < (f_i(x) - x_i)/2$, and $z \in A_i(y)$. Then

$$\begin{aligned} |f_i(y) - f_i(z)| &\leqslant \|f(y) - f(z)\| \leqslant \|y - z\| = \max(y_i - z_i, |y_j - z_j|) = y_i - z_i, \\ f_i(y) - y_i &= f_i(x) - (f_i(x) - f_i(y)) - x_i - (y_i - x_i) \\ &\geqslant f_i(x) - x_i - 2\|y - x\| > f_i(x) - x_i - (f_i(x) - x_i) = 0, \end{aligned}$$

and

$$f_i(z) = f_i(y) + (f_i(z) - f_i(y)) > y_i - (y_i - z_i) = z_i.$$

To show (ii) we take y with $\|y-x\| < (x_i - f_i(x))/2$, and $z \in B_i(y)$. Then

$$|f_i(z)-f_i(y)| \leqslant \|f(z)-f(y)\| \leqslant \|z-y\| = \max(z_i - y_i, |z_j - y_j|) = z_i - y_i,$$
$$\begin{aligned} f_i(y)-y_i &= f_i(x)+(f_i(y)-f_i(x))-x_i+(x_i-y_i) \\ &\leqslant f_i(x)-x_i+2\|y-x\| < f_i(x)-x_i+(x_i-f_i(x)) = 0, \end{aligned}$$

and

$$f_i(z) = f_i(y)+(f_i(z)-f_i(y)) < y_i+(z_i-y_i) = z_i.$$

∎

Figure 2 illustrates the "envelope" sets $A_1(x)$, $A_2(x)$, $B_1(x)$, and $B_2(x)$ defined in Theorem 3.1. Henceforth we assume the grid directions shown in this figure, that is, for $x \in D_{a,b}$, x_1 increases to the right and x_2 increases upward.

The following corollary shows that if $x \in D_{a,b}$ is in $F_1(f)$ or $F_2(f)$ but $x \notin F_1(f) \cap F_2(f)$, then we can choose an envelope set of x that is guaranteed to contain a fixed point of f.

Corollary 3.2. *Let (i, j) be a permutation of $(1, 2)$. Take $x \in F_i(f)$ such that $f_j(x) < x_j$ (resp. $f_j(x) > x_j$), and suppose there exists $y \in F(f)$ such that $y \notin A_j(x)$ (resp. $y \notin B_j(x)$). Let E be the closed edge in $\partial A_j(x)$ (resp. $\partial B_j(x)$) such that for every $e \in E$, the open line segment with endpoints at e and y does not intersect $A_j(x)$ (resp. $B_j(x)$). Define $Z \equiv Z(x, y) \equiv \{z \in E : |z_j - y_j| \leqslant |z_i - y_i|\}$. Then $F(f) \cap Z \neq \varnothing$. We conclude that $F(f) \cap A_j(x) \neq \varnothing$ (resp. $F(f) \cap B_j(x) \neq \varnothing$).*

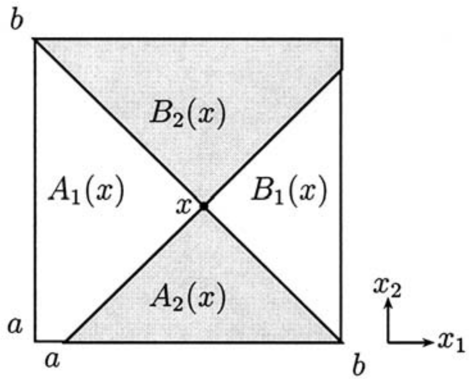


FIG. 2. Envelopes $A_1(x)$, $A_2(x)$, $B_1(x)$, and $B_2(x)$.

Proof. See Fig. 3 for an illustration of the case $(i, j) = (1, 2)$ and $f_2(x) < x_2$. The hypothesis $f_j(x) < x_j$ (resp. $f_j(x) > x_j$) implies that $F_j(f) \cap B_j(x) = \varnothing$ (resp. $F_j(f) \cap A_j(x) = \varnothing$), so clearly $y \notin B_j(x)$ (resp. $y \notin A_j(x)$) and Z is well defined with $Z \neq \varnothing$. Suppose that $Z \cap F_j(f) = \varnothing$. Since $x \in Z$ and Z is connected, by Lipschitz continuity it must be true that $f_j(z) < z_j$ (resp. $f_j(z) > z_j$) for every $z \in Z$. This implies that $m = \min\{z_j : z \in Z\} > a$ (resp. $m \equiv \max\{z_j : z \in Z\} < b$). From this and the definition of Z it follows that if $z' \in Z$ is such that $z'_j = m$, then $|z'_j - y_j| = |z'_i - y_i|$. Since $f_j(z') < z'_j$ and $y \in B_j(z')$ (resp. $f_j(z') > z'_j$ and $y \in A_j(z')$), $y \notin F_j(f)$, a contradiction. Hence there exists $z \in Z \cap F_j(f)$. If $z \notin F_i(f)$ then by Theorem 3.1 either $x \notin F_i(f)$ or $y \notin F_i(f)$, since $|z_j - y_j| \leqslant |z_i - y_i|$. Both cases are contradictions, so $z \in F(f)$. ∎

The following corollary shows that if $x \in D_{a,b}$ is in $F_1(f)$ or $F_2(f)$ but not both, $y \in D_{a,b}$ is a fixed point of f, and both x and y are contained in a closed rectangle D' whose sides have slope 1 or -1, then we can choose a certain envelope set of x whose intersection with D' is guaranteed to contain a fixed point of f.

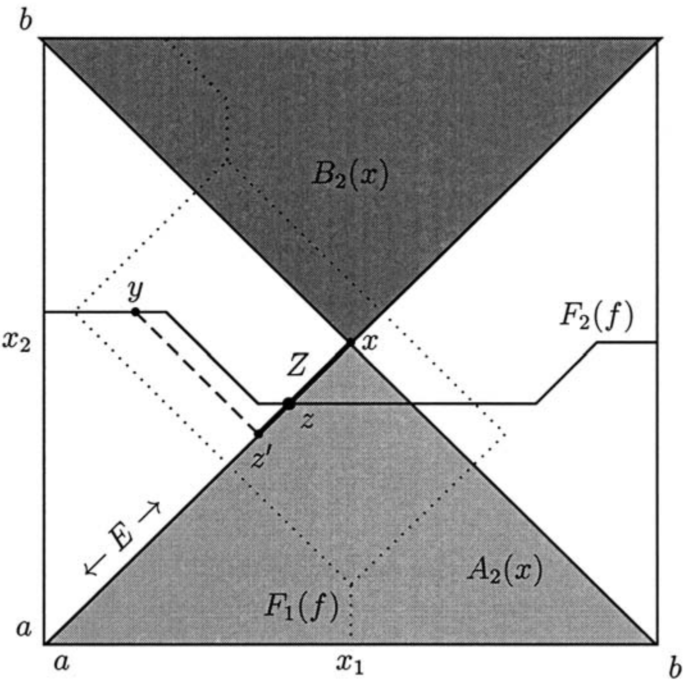


FIG. 3. Corollary 3.2. $f_2(x) < x_2$, $y \in F(f)$, and $F(f)$ intersects Z at z.

COROLLARY 3.3. *Let (i, j) be a permutation of $(1,2)$. Let $D' \subset D_{a,b}$ be a closed rectangle whose sides have slope 1 or -1, and suppose there exists $y \in D' \cap F(f)$. Let $x \in D' \cap F_i(f)$ be such that $f_j(x) < x_j$ (resp. $f_j(x) > x_j$). Then $D' \cap A_j(x) \cap F(f) \neq \varnothing$ (resp. $D' \cap B_j(x) \cap F(f) \neq \varnothing$).*

Proof. If $y \in A_j(x)$ (resp. $y \in B_j(x)$) then the conclusion holds, so assume that $y \notin A_j(x)$ (resp. $y \notin B_j(x)$). Then the set $Z \equiv Z(x, y)$ defined in Corollary 3.2 is contained in D'. The remainder follows from Corollary 3.2. ∎

3.2. *Extension Theorem*

THEOREM 3.4. *Given $a', b' \in \mathbb{R}$ with $a' < a$ and $b' > b$, and a function $f \in \mathfrak{F}_{a,b}$, we define an extended function $\bar{f}\colon D_{a',b'} \to D_{a,b}$ as $\bar{f}(x) = f(P(x))$, where the projection P is given by*

$$P(x) \equiv (\max(a, \min(b, x_1)), \max(a, \min(b, x_2))).$$

The following are true:

- *The function $\bar{f}$ is Lipschitz continuous with constant 1 on $D_{a',b'}$, $\bar{f}$ has at least one fixed point, and $D_{a,b}$ contains all fixed points of $\bar{f}$.*
- *Let $\tilde{y} \in D_{a',b'}$, be such that $\|\bar{f}(\tilde{y}) - \tilde{y}\| \leqslant \varepsilon$. Then $\|f(\tilde{x}) - \tilde{x}\| \leqslant \varepsilon$, where $\tilde{x} = P(\tilde{y})$.*
- *Let $\tilde{y} \in D_{a',b'}$ be such that $\|\tilde{y} - \alpha\| \leqslant \varepsilon$ for some fixed point α of $\bar{f}$. Then $\|\tilde{x} - \alpha\| \leqslant \varepsilon$ where $\tilde{x} = P(\tilde{y})$.*

Proof. For all $x, y \in D_{a',b'}$,

$$\|\bar{f}(x) - \bar{f}(y)\| \leqslant \|P(x) - P(y)\| \leqslant \|x - y\|$$

so $\bar{f}$ is Lipschitz continuous with constant 1 on $D_{a',b'}$. Since the range of $\bar{f}$ is $D_{a,b} \subset D_{a',b'}$, by the Brouwer theorem $\bar{f}$ has at least one fixed point, and obviously $D_{a,b}$ contains all fixed points of $\bar{f}$. Finally,

$$\varepsilon \geqslant \|\bar{f}(\tilde{y}) - \tilde{y}\| = \|f(\tilde{x}) - \tilde{y}\| \geqslant \|f(\tilde{x}) - \tilde{x}\|$$

and

$$\varepsilon \geqslant \|\tilde{y} - \alpha\| \geqslant \|\tilde{x} - \alpha\|$$

since $\tilde{x} \in D_{a,b}$, $f(\tilde{x}) \in D_{a,b}$, $\alpha \in D_{a,b}$, and $\tilde{y} \in D_{a,b}$ implies $\tilde{y} = \tilde{x}$. ∎

3.3. *Termination Theorem*

Given a closed rectangle R whose sides have slope 1 or -1, we define $l_1(R)$ (resp. $l_{-1}(R)$) as the length of a side of R with slope 1 (resp. -1). In addition we define $l_{\min}(R) \equiv \min(l_1(R), l_{-1}(R))$ and $l_{\max}(R) \equiv \max(l_1(R), l_{-1}(R))$. We define $c(R) \equiv (c_1(R), c_2(R))$ as the center of R. Obviously, a line with slope $s \in \{1, -1\}$ through $c(R)$ divides R into two halves R_1, R_2 such that $l_{-s}(R_1) = l_{-s}(R_2) = l_{-s}(R)/2$.

The following theorem establishes conditions under which $c(R)$ satisfies the absolute or residual criterion.

THEOREM 3.5. *Let $R \subset D_{a,b}$ be a rectangle as above, such that*

$$l_1(R) + l_{-1}(R) \leqslant \sqrt{2}\,\varepsilon \tag{5}$$

and both $F_1(f)$ and $F_2(f)$ intersect R. Then $\tilde{y} = c(R)$ satisfies (2). *If, in addition, R contains a fixed point of f, then $\tilde{y}$ satisfies* (3).

Proof. The hypothesis implies that the smallest square containing R has sides of length $\leqslant \varepsilon$ (see Fig. 4). Since R intersects both $F_1(f)$ and $F_2(f)$, there exist $x^1 \in R \cap F_1(f)$ and $x^2 \in R \cap F_2(f)$ such that $\|c(R) - x^1\| \leqslant \varepsilon/2$ and $\|c(R) - x^2\| \leqslant \varepsilon/2$. Then for every $i \in \{1, 2\}$,

$$|f_i(c(R)) - c_i(R)| = |f_i(c(R)) - f_i(x^i) + x_i^i - c_i(R)| \leqslant 2\|x^i - c(R)\| \leqslant \varepsilon,$$

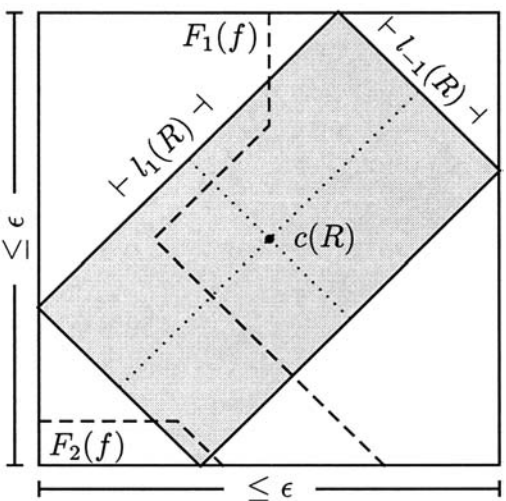


FIG. 4. R satisfies (5) so $c(R)$ satisfies (2).

where $c(R) \equiv (c_1(R), c_2(R))$. It follows that $\|f(c(R)) - c(R)\| \leqslant \varepsilon$. If R contains a fixed point α of f then (3) trivially follows from the fact that R is contained in a square with sides of length ε; indeed, we have $\|\tilde{y} - \alpha\| \leqslant \varepsilon/2$. ▮

4. THE BEFix ALGORITHM

In this section we provide necessary definitions, describe the algorithm, and list its pseudocode.

4.1. *Definitions*

We define the domains $\bar{D} \equiv D_{-0.5,\,1.5}$ and $D_0 \equiv \{x \in \mathbb{R}^2 : \|x - (0.5, 0.5)\|_1 \leqslant 1\}$. Observe that D_0 is the smallest 1-norm ball containing D and the largest contained in $\bar{D}$ (see Fig. 5).

Given a function $f \in \mathfrak{F}$ we define its extension $\bar{f}\colon \bar{D} \to D$ as in Theorem 3.4. By this theorem, $\bar{f}$ is Lipschitz continuous with constant 1 on $\bar{D}$ and has at least one fixed point, and D contains all of its fixed points. Hence, if $\tilde{y}$ is a residual solution for $\bar{f}$ then $\tilde{x} = P(\tilde{y})$ is a residual solution for f, where P projects $\bar{D}$ onto D, i.e.,

$$P(x) \equiv (\max(0, \min(1, x_1)), \max(0, \min(1, x_2))).$$

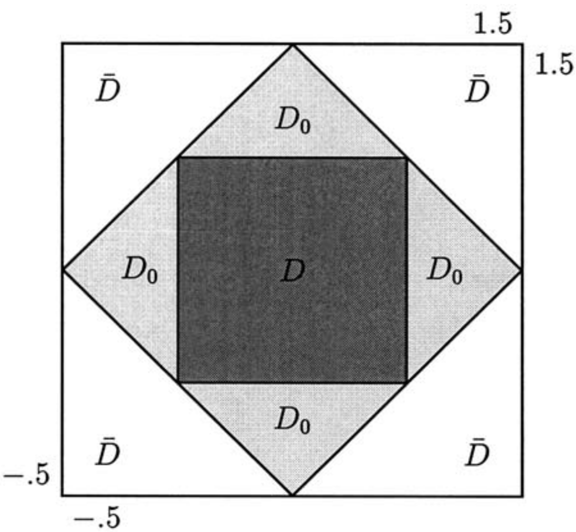


FIG. 5. D, D_0, and $\bar{D}$.

4.2. *Description*

The algorithm computes $\tilde{y} \in D_0$ satisfying the residual criterion

$$\|\bar{f}(\tilde{y}) - \tilde{y}\| \leqslant \varepsilon \tag{6}$$

and takes $\tilde{x} = P(\tilde{y})$ as a solution to (2). In addition it returns a logical variable *abs* which is true only if $\tilde{y}$ (and by extension $\tilde{x}$) also satisfies the absolute criterion

$$\|\tilde{y} - \alpha\| \leqslant \varepsilon \tag{7}$$

for some fixed point α of $\bar{f}$.

Step k of the algorithm ($k \geqslant 1$) evaluates $\bar{f}$ at x^k, the center of the rectangle D_{k-1}, and constructs a rectangle $D_k \subset D_{k-1}$ by bisecting D_{k-1} along lines with slope 1 or -1 through x^k. Each D_k is a closed rectangle whose sides have slope 1 or -1 (this is clearly true of D_0). The algorithm terminates at step k if a residual criterion is satisfied at x^k, or if $l_1(D_k) + l_{-1}(D_k) \leqslant \sqrt{2}\,\varepsilon$ so that the center of D_k satisfies the residual criterion by Theorem 3.5. At each step D_k is guaranteed to contain a fixed point of $\bar{f}$, so if D_k satisfies (5) then the center of D_k also satisfies the absolute criterion.

4.3. *Pseudocode*

```
k := 0;
repeat
   k := k + 1;
   D_k := D_{k-1};
   x^k := c(D_k);
   v_1 := \bar{f}_1(x^k) - x^k_1;
   v_2 := \bar{f}_2(x^k) - x^k_2;
   if v_1 = 0 and v_2 = 0 then
      terminate with \tilde{x} = P(x^k), abs = true;
   end(if)
   if max(|v_1|, |v_2|) <= \varepsilon then
      terminate with \tilde{x} = P(x^k), abs = false;
   end(if)
   ! Condition 1
   if l_min(D_{k-1}) > (\sqrt{2}/2) \varepsilon then
      if v_1 > 0 and v_2 > 0 then
         D_k := D_k \cap (B_1(x^k) \cup B_2(x^k));
      else if v_1 < 0 and v_2 > 0 then
         D_k := D_k \cap (A_1(x^k) \cup B_2(x^k));
```

else if $v_1 < 0$ and $v_2 < 0$ then
$D_k := D_k \cap (A_1(x^k) \cup A_2(x^k))$;
else if $v_1 > 0$ and $v_2 < 0$ then
$D_k := D_k \cap (B_1(x^k) \cup A_2(x^k))$;
else if $v_1 = 0$ and $v_2 < 0$ then
$D_k := D_k \cap A_2(x^k)$;
else if $v_1 = 0$ and $v_2 > 0$ then
$D_k := D_k \cap B_2(x^k)$;
else if $v_1 < 0$ and $v_2 = 0$ then
$D_k := D_k \cap A_1(x^k)$;
else if $v_1 > 0$ and $v_2 = 0$ then
$D_k := D_k \cap B_1(x^k)$;
end(if)
! Condition 2
if $l_1(D_{k-1}) \leqslant \sqrt{2}\,\varepsilon$ and $l_{-1}(D_{k-1}) > (\sqrt{2}/2)\,\varepsilon$ then
if $v_1 < -\varepsilon$ or $v_2 > \varepsilon$ then
$D_k := D_k \cap (A_1(x^k) \cup B_2(x^k))$;
else
$D_k := D_k \cap (B_1(x^k) \cup A_2(x^k))$;
end(if)
! Condition 3
if $l_{-1}(D_{k-1}) \leqslant \sqrt{2}\,\varepsilon$ and $l_1(D_{k-1}) < (\sqrt{2}/2)\,\varepsilon$ then
if $v_1 > \varepsilon$ or $v_2 > \varepsilon$ then
$D_k := D_k \cap (B_1(x^k) \cup B_2(x^k))$;
else
$D_k := D_k \cap (A_1(x^k) \cup A_2(x^k))$;
end(if)
end(if)
until $l_1(D_k) + l_{-1}(D_k) \leqslant \sqrt{2}\,\varepsilon$;
terminate with $\tilde{x} = P(c(D_k))$, $abs =$ true;

5. ALGORITHM ANALYSIS

In this section we analyze the algorithm and show that it computes $\tilde{y}$ satisfying (6) in at most $2\lceil \log_2(1/\varepsilon) \rceil + 1$ loop iterations. Each iteration evaluates $\bar{f}$ (equivalently f) once, so we obtain the desired complexity (4).

The algorithm assumes upon entry to step k that D_{k-1} contains a fixed point of $\bar{f}$; clearly this is true for D_0. It sets $x^k = c(D_{k-1})$ and computes $v_1 \equiv \bar{f}_1(x^k) - x_1^k$ and $v_2 \equiv \bar{f}_2(x^k) - x_2^k$. If $\max(|v_1|, |v_2|) \leqslant \varepsilon$ then x^k satisfies (6) (and (7) if $v_1 = v_2 = 0$) and the algorithm terminates with $\tilde{x} = P(x^k)$.

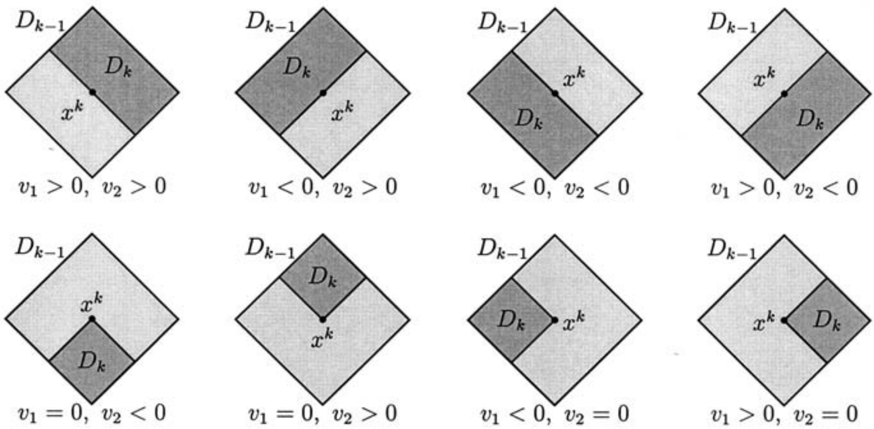


FIG. 6. Condition 1: Choice of D_k at step k.

Otherwise the algorithm sets D_k to D_{k-1} and tests the following three conditions in order, reducing D_k by the associated rules for each condition that holds true. These conditions and rules ensure by Theorem 3.1 and Corollary 3.3 that each D_k contains a fixed point of $\bar{f}$. The rules for Condition 1 reduce D_k to a set containing a fixed point, and the rules for Conditions 2 and 3 eliminate from D_k only subsets that do not contain a fixed point, so D_k cannot become empty.

Condition 1. $l_{\min}(D_{k-1}) > (\sqrt{2}/2)\,\varepsilon$.

See Fig. 6 for an illustration of the following rules.

- If $v_1 > 0$ and $v_2 > 0$ then $D_k = D_k \cap (B_1(x^k) \cup B_2(x^k))$.
- If $v_1 < 0$ and $v_2 > 0$ then $D_k = D_k \cap (A_1(x^k) \cup B_2(x^k))$.
- If $v_1 < 0$ and $v_2 < 0$ then $D_k = D_k \cap (A_1(x^k) \cup A_2(x^k))$.
- If $v_1 > 0$ and $v_2 < 0$ then $D_k = D_k \cap (B_1(x^k) \cup A_2(x^k))$.
- If $v_1 = 0$ and $v_2 < 0$ then $D_k = D_k \cap A_2(x^k)$.
- If $v_1 = 0$ and $v_2 > 0$ then $D_k = D_k \cap B_2(x^k))$.
- If $v_1 < 0$ and $v_2 = 0$ then $D_k = D_k \cap A_1(x^k)$.
- If $v_1 > 0$ and $v_2 = 0$ then $D_k = D_k \cap B_1(x^k)$.

Condition 2. $l_1(D_{k-1}) \leqslant \sqrt{2}\,\varepsilon$ and $l_{-1}(D_{k-1}) > (\sqrt{2}/2)\,\varepsilon$.

We define the points

$$w^1 \equiv x^k - \left(\frac{\sqrt{2}}{4} l_1(D_{k-1}), \frac{\sqrt{2}}{4} l_1(D_{k-1}) \right),$$
$$w^2 \equiv x^k + \left(\frac{\sqrt{2}}{4} l_1(D_{k-1}), \frac{\sqrt{2}}{4} l_1(D_{k-1}) \right)$$

(see Fig. 7); w^1 and w^2 are the midpoints of the edges of D_{k-1} with slope -1. If $v_1 > \varepsilon$ then since $\|w^2 - x^k\| \leqslant \varepsilon/2 < v_1/2$, by Theorem 3.1 $A_1(w^2) \cap F_1(\bar{f}) = \varnothing$. Similarly, if $v_1 < -\varepsilon$ then $B_1(w^1) \cap F_1(\bar{f}) = \varnothing$; if $v_2 > \varepsilon$ then $A_2(w^2) \cap F_1(\bar{f}) = \varnothing$; if $v_2 < -\varepsilon$ then $B_2(w^1) \cap F_1(\bar{f}) = \varnothing$. Hence we obtain the following rules.

- If $v_1 < -\varepsilon$ or $v_2 > \varepsilon$ then $D_k := D_k \cap (A_1(x^k) \cup B_2(x^k))$.
- If $v_1 < \varepsilon$ or $v_2 < -\varepsilon$ then $D_k := D_k \cap (A_2(x^k) \cup B_1(x^k))$.

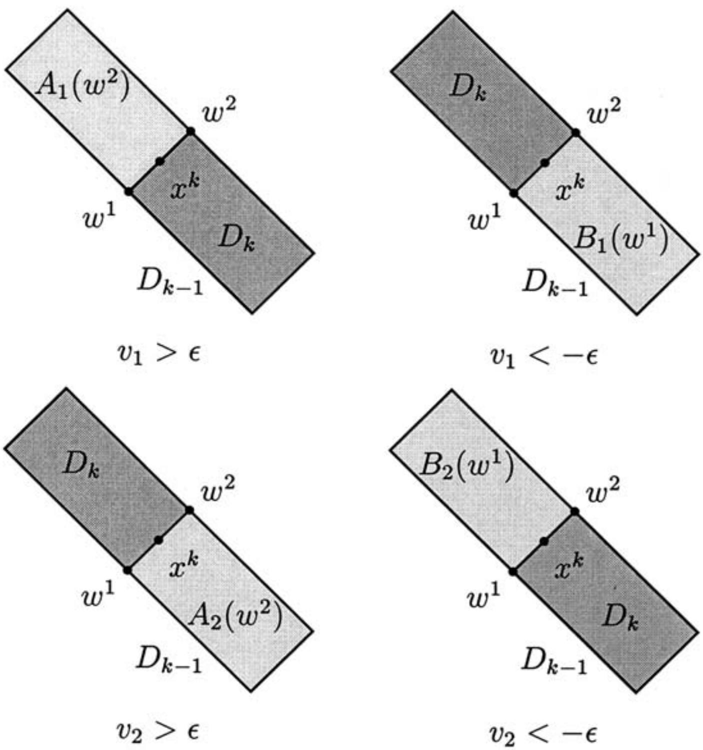


FIG. 7. Condition 2: Choice of D_k at step k.

Condition 3. $l_{-1}(D_{k-1}) \leqslant \sqrt{2}\,\varepsilon$ and $l_1(D_{k-1}) > (\sqrt{2}/2)\,\varepsilon$.

We define the points

$$w^1 \equiv x^k + \left(-\frac{\sqrt{2}}{4} l_{-1}(D_{k-1}), \frac{\sqrt{2}}{4} l_{-1}(D_{k-1})\right),$$

$$w^2 \equiv x^k + \left(\frac{\sqrt{2}}{4} l_{-1}(D_{k-1}), -\frac{\sqrt{2}}{4} l_{-1}(D_{k-1})\right)$$

(see Fig. 8); w^1 and w^2 are the midpoints of the edges of D_{k-1} with slope 1. If $v_1 > \varepsilon$ then since $\|w^2 - x^k\| \leqslant \varepsilon/2 < v_1/2$, by Theorem 3.1, $A_1(w^2) \cap F_1(\bar{f}) = \varnothing$. Similarly, if $v_1 < -\varepsilon$ then $B_1(w^1) \cap F_1(\bar{f}) = \varnothing$; if $v_2 > \varepsilon$ then $A_2(w^1) \cap F_1(\bar{f}) = \varnothing$; if $v_2 < -\varepsilon$ then $B_2(w^2) \cap F_1(\bar{f}) = \varnothing$. Hence we obtain the following rules.

- If $v_1 > \varepsilon$ or $v_2 > \varepsilon$ then $D_k := D_k \cap (B_1(x^k) \cup B_2(x^k))$.
- If $v_1 > \varepsilon$ or $v_2 < -\varepsilon$ then $D_k := D_k \cap (A_1(x^k) \cup A_2(x^k))$.

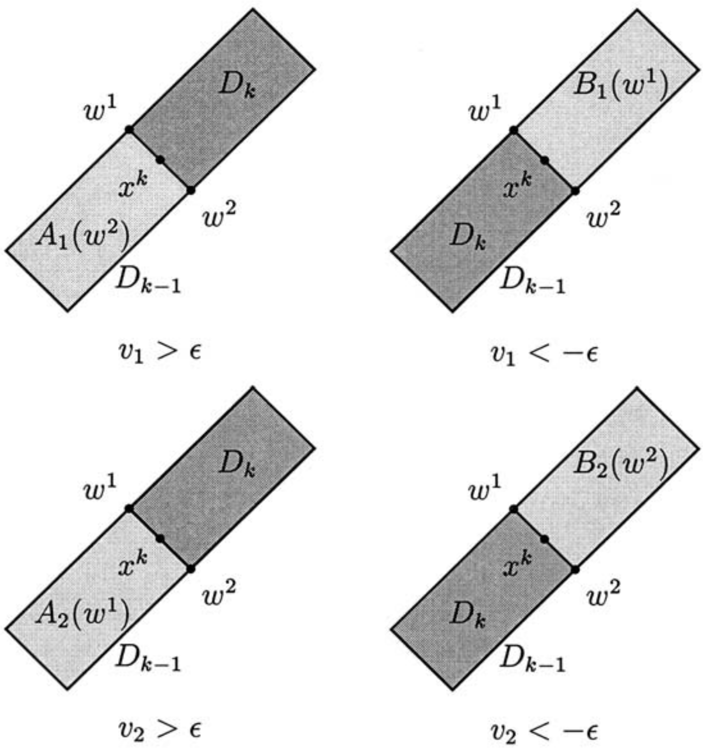


FIG. 8. Condition 3: Choice of D_k at step k.

5.1. *Termination*

At the end of step k the algorithm terminates with $\tilde{x} = P(c(D_k))$ if D_k satisfies

$$l_1(D_k) + l_{-1}(D_k) \leqslant \sqrt{2}\,\varepsilon, \tag{8}$$

or proceeds to the next step otherwise. By Theorem 3.5, if D_k satisfies (8) then $\tilde{y} = c(D_k)$ satisfies (6) and (7).

5.2. *Complexity*

The algorithm ensures that if $l_{\min}(D_{k-1}) > (\sqrt{2}/2)\,\varepsilon$ at step k then $l_s(D_k) = l_s(D_{k-1})/2$ for some $s \in \{1, -1\}$; otherwise, $l_s(D_{k-1}) \leqslant (\sqrt{2}/2)\,\varepsilon$ for some $s \in \{1, -1\}$, and $l_{-s}(D_m) = l_{-s}(D_{m-1})/2$ for all $m \geqslant k$.

Suppose that when the algorithm is applied to a function f, exactly $\bar{k} = \bar{k}(f)$ steps are required to compute $D_{\bar{k}}$ satisfying $l_{\max}(D_{\bar{k}}) \leqslant (\sqrt{2}/2)\,\varepsilon$. We show that under this condition at least one step $k \leqslant \bar{k}$ of the algorithm must reduce by half both $l_1(D_k)$ and $l_{-1}(D_k)$. Indeed, for at least one step k it must be true that $l_{\min}(D_{k-1}) > (\sqrt{2}/2)\,\varepsilon$ and for some $s \in \{1, -1\}$, $l_s(D_k) \leqslant (\sqrt{2}/2)\,\varepsilon$ (so that $l_s(D_{k-1}) \leqslant \sqrt{2}\,\varepsilon$). If $v_1 = v_2 = 0$ then the algorithm terminates. Otherwise, if either $v_1 = 0$ or $v_2 = 0$ at step k then $l_s(D_k) = l_s(D_{k-1})/2$ for both $s = 1$ and $s = -1$, by Condition 1. If $v_1 \neq 0$ and $v_2 \neq 0$ then, since $l_s(D_{k-1}) \leqslant \sqrt{2}\,\varepsilon$, Conditions 2 and 3 ensure that $l_{-s}(D_k) = l_{-s}(D_{k-1})/2$.

Since $l_1(D_0) = l_{-1}(D_0) = \sqrt{2}$, D_k will satisfy $l_{\max}(D_k) \leqslant (\sqrt{2}/2)\,\varepsilon$, and thus (8), while

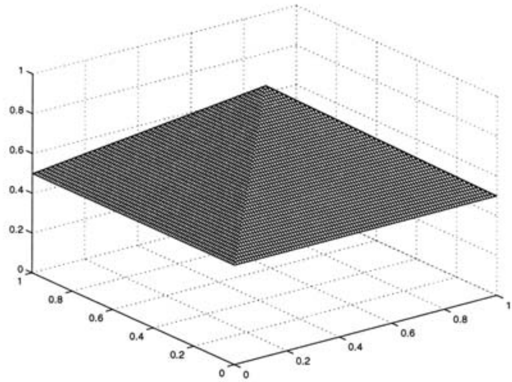
$$k \leqslant 2\lceil \log_2 (2/\varepsilon) \rceil - 1 = 2\lceil \log_2 (1/\varepsilon) \rceil + 1.$$

6. NUMERICAL TESTS

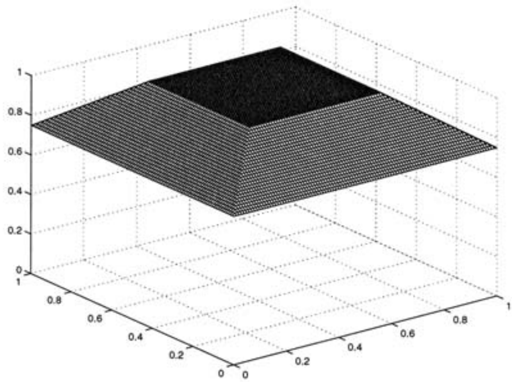
We tested our algorithm on a number of pyramid functions, defined as follows. For $b \in D$ and $h \in [0, 1]$ we define the pyramid basis function $P_b^h\colon D \to [0, 1]$ as

$$P_b^h(x) \equiv \min(1, \max(h - \|x - b\|_\infty, 0)). \tag{9}$$

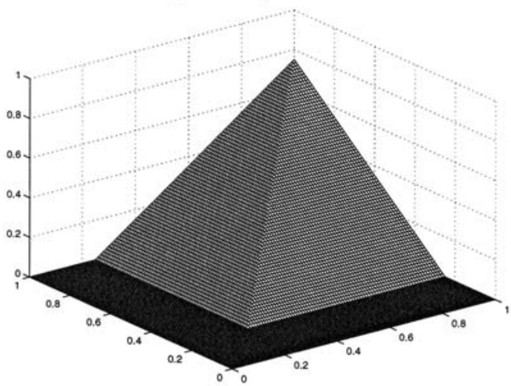
Figure 9 shows plots of P_b^h for several values of b and h.



$b = (0.5, 0.5), \ h = 1$



$b = (0.5, 0.5), \ h = 1.25$



$b = (1, 1), \ h = .75$

FIG. 9. The pyramid basis function P_b^h.

Our test functions are based on the following pyramid basis functions:

- $b_1 = (0, 0)$, $h_1 = 1$.
- $b_2 = (0, 1)$, $h_2 = 1.1$.
- $b_3 = (1, 0)$, $h_3 = 0.9$.
- $b_4 = (1, 1)$, $h_4 = 1.01$.
- $b_5 = (0, 0.5)$, $h_5 = 1$.
- $b_6 = (0.5, 0.5)$, $h_6 = 0.8$.
- $b_7 = (0.6, 0.4)$, $h_7 = 1.2$.
- $b_8 = (0.4, 0.6)$, $h_8 = 0.9$.
- $b_9 = (0.6, 0.98)$, $h_9 = 0.99$.
- $b_{10} = (0.98, 0.3)$, $h_{10} = 0.99$.
- $b_{11} = (0.27, 0.64)$, $h_{11} = 1.01$.
- $b_{12} = (0.64, 0.27)$, $h_{12} = 0.99$.
- $b_{13} = (0, 0)$, $h_{13} = 0.1$.

Given the distinct integers $i_1, \ldots, i_k$, $1 \leqslant k \leqslant 13$, $1 \leqslant i_j \leqslant 13 \forall j$, we define the pyramid function

$$P_{i_1, \ldots, i_k}(x) \equiv \max(p_{b_{i_1}}^{h_{i_1}}(x), \ldots, P_{b_{i_k}}^{h_{i_k}}(x)). \tag{10}$$

Figure 10 shows the plot of $P_{10, 11, 12}(x)$.

We tested the algorithm on the functions $f(x) = (P_{S_1}(x), P_{S_2}(x))$ for all pairs of nonempty subsets S_1 and S_2 of $\{1, \ldots, 13\}$, and with $\varepsilon = 0.0001$. The tests yielded the following statistics:

- Total number of tests: 67, 092, 481 (or $(2^{13} - 1)^2$).
- Total number of tests satisfying the absolute error criterion: 21,776.
- Average ratio of a test's function evaluations to $2\lceil \log_2(1/\varepsilon) \rceil + 1 = 29 : 0.759$.
- Average ratio of a test's function evaluations to $2\lceil \log_2(1/\varepsilon) \rceil + 1 = 29$, for tests satisfying the absolute error criterion: 0.522.
- Minimum number of function evaluations achieved by a test: 1.
- Average ratio of the length of a longer edge of D_k to the edge length of D_0, where k is the final step of a test: 0.342.
- Number of tests requiring exactly $2\lceil \log_2(1/\varepsilon) \rceil + 1 = 29$ function evaluations: 4,195,852.

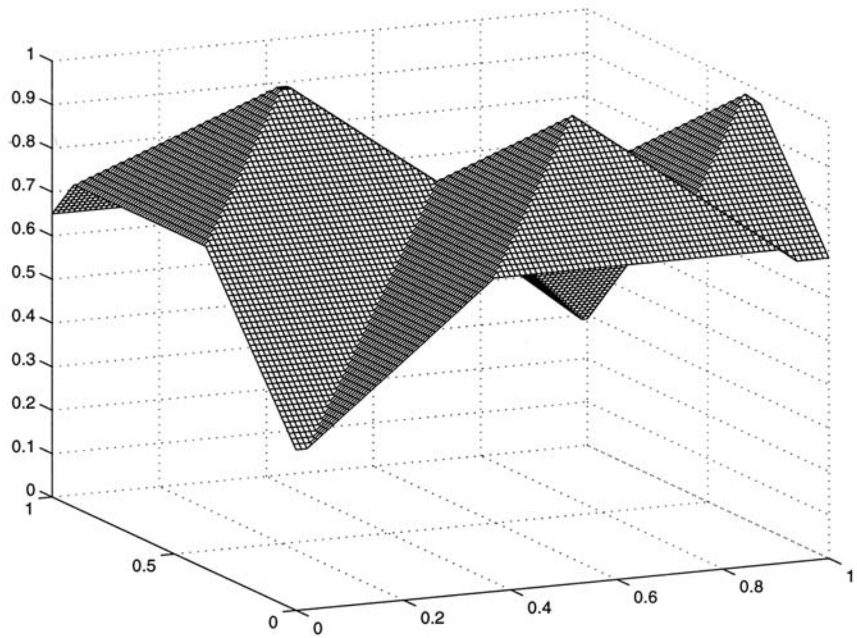


FIG. 10. The pyramid function $P_{10,11,12}$.

7. FUTURE WORK

We plan to develop a version of our algorithm that works for any dimension $d \geqslant 2$. We believe that this extended algorithm will have optimal worst-case complexity in the class of algorithms that use information consisting of function evaluations. We conjecture that the complexity of the algorithm and the lower bound are $\mathcal{O}(c(d) \log(1/\varepsilon))$, where $c(d)$ is a polynomial in d. The average case complexity should also be explored.

We believe that the absolute criterion problem, that is, the computation of $\tilde{x} \in D_d$ satisfying

$$\|\tilde{x} - \alpha\| \leqslant \varepsilon, \tag{11}$$

where $\alpha \in D_d$ is a fixed point of f, has infinite worst-case complexity when information consists of function evaluations. We plan to investigate restricted function classes that may make the complexity finite in the absolute criterion case.

REFERENCES

1. G. Allgower and K. Georg, "Numerical Continuation Methods," Springer-Verlag, New York/Berlin, 1990.
2. B. C. Eaves, Homotopies for computation of fixed points, *Math. Programming* (1972), 1–22.
3. B. C. Eaves and R. Saigal, Homotopies for computation of fixed points on unbounded regions, *Math. Programming* (1972), 225–237.
4. M. D. Hirsch, C. Papadimitriou, and S. Vavasis, Exponential lower bounds for finding brouwer fixed points, *J. Complexity* **5** (1989), 379–416.
5. Z. Huang, L. Khachiyan, and K. Sikorski, Approximating fixed points of weakly contracting mappings, *J. Complexity* **15** (1999), 200–213.
6. A. Nemirovsky, private communication, 1999.
7. A. Nemirovsky and D. B. Yudin, "Problem Complexity and Method Efficiency in Optimization," Wiley, New York, 1983.
8. M. Scarf, The approximation of fixed points of a continuous mapping, *SIAM J. Appl. Math.* (1967), 1328–1343.
9. K. Sikorski, "Optimal solution of Nonlinear Equations," Oxford Press, New York, 2001.
10. K. Sikorski, C. W. Tsay, and H. Woźniakowski, An ellipsoid algorithm for the computation of fixed points, *J. Complexity* **9** (1993), 181–200.