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On the Existence of a Perfect Matching for 4-regular Graphs derived from Quadrilateral Meshes

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Abstract:

In 1891, Peterson [Pet91] proved that every 3-regular bridgeless graph has a perfect matching. It is well-known that the dual of a triangular mesh on a compact manifolds is a 3-regular graph. M. Gopi and D. Eppstein [GE04] use Petersons theorem to solve the problem of constructing strips of triangles from triangular meshes on a compact manifold. P. Diaz-Gutierrez and M. Gopi [DG04] elaborate on the creation of strips of quadrilaterals when a perfect matching exists.
In this paper, it is shown that the dual of a quadrilateral mesh on a 2-dimensional compact manifold with an even number of quadrilaterals (which is a 4-regular graph) also has a perfect matching. In general, however, not all 4-regular graphs have a perfect matching. Indeed, a counter-example is given that is planar.

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In 1891, Peterson [Pet91] proved that every 3-regular bridgeless graph has a perfect matching. It is well-known that the dual of a triangular mesh on a compact manifolds is a 3-regular graph. M. Gopi and D. Eppstein [GE04] use Peterson's theorem to solve the problem of constructing strips of triangles from triangular meshes on a compact manifold. P. Diaz-Gutierrez and M. Gopi [DG04] elaborate on the creation of strips of quadrilaterals when a perfect matching exists.

In this paper, it is shown that the dual of a quadrilateral mesh on a 2-dimensional compact manifold with an even number of quadrilaterals (which is a 4-regular graph) also has a perfect matching. In general, however, not all 4-regular graphs have a perfect matching. Indeed, a counter-example is given that is planar.

Introduction

In 1891, Peterson [Pet91] showed that every 3-regular graph has a perfect matching. Recent attention has highlighted this result, and M. Gopi and D. Eppstein [GE04] use Peterson's theorem to solve the problem of constructing strips of triangles from triangular meshes on a compact manifold. Additionally, P. Diaz-Gutierrez and M. Gopi [DG04] elaborate on the creation of strips of quadrilaterals when a perfect matching exists. They conjecture that every quadrilateral mesh with an even number of elements has a perfect matching, but do not supply a proof.

In this paper, Tutte's Theorem will be used to prove that a 4-regular graph that is the dual of a quadrilateral mesh on a 2-dimensional compact manifold with an even number of quadrilaterals has a perfect match. An example of a 4-regular planar graph is given that does not have a perfect match, and, thereby also providing an example of a planar graph that is not dual to a quadrilateral mesh.

Theorem (Tutte [Tu47])

A graph **G** has a perfect match if and only if, for every subset **S** of the vertices of the graph **G**, the number of connected components in **G-S** with an odd number of nodes is less than or equal to the number of vertices in S.

In general, not all 4-regular graphs have a perfect match as Figure 1 illustrates. In this example, S has two vertices. G-S has 4 connected components; three of them have an odd number of vertices violating Tutte's condition.

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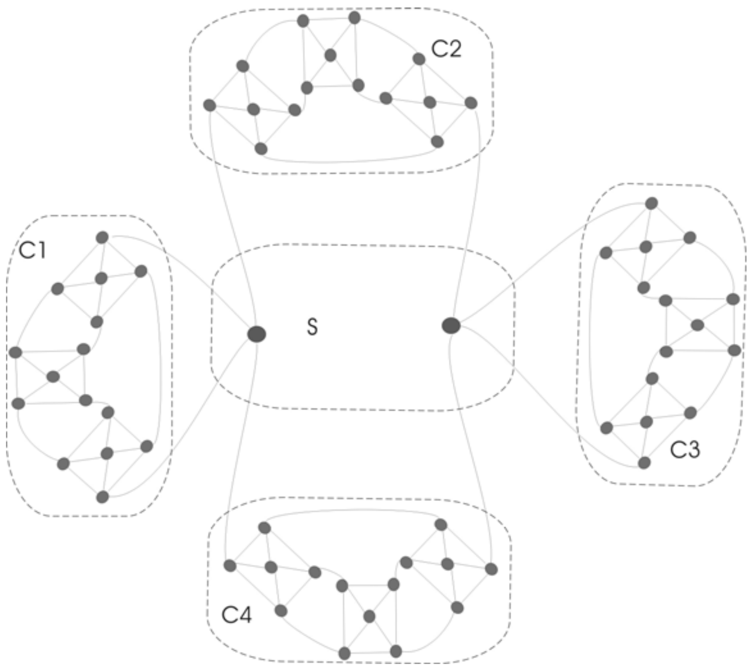

Figure 1

Mesh Definitions and Assumptions

Def. 1 A quadrilateral mesh **M** is a geometric *cell complex* [3] composed of 0-Dimensional nodes **N**, 1-dimensional edges **E**, and 2-dimensional quadrilaterals **Q** such that

1. Each node is contained by at least two edges.
2. Each edge contains two distinct nodes.
3. If two edges contain the same nodes, the edges are identical.
4. Each quadrilateral is bounded by a cycle of four distinct edges.
5. Two nodes have at most one edge between them.
6. Two quadrilaterals share at most one edge [1].
7. Every edge is contained by no more than two quadrilaterals, and not less than one.
8. If two quadrilaterals share four nodes, they are identical.

When dealing with a quadrilateral mesh on a compact 2-manifold in $\Re^3$, the following requirement is added:

9. Every edge is contained by exactly two distinct quadrilaterals.

Def. 2 The boundary of any quadrilateral mesh is the collection of edges that are contained by only one quadrilateral.

Def. 3 Internal edges are edges that are not contained by the boundary.

Prop 1. Every node in the boundary of a mesh is connected to an even number of boundary edges.

Proof- let n be a node in the mesh **M**. Let $\boldsymbol{i}$ be the number of interior edges that contain n, $\boldsymbol{b}$ be the number of boundary edges that contain n, and $\boldsymbol{q}$ the number of quadrilaterals that contain n.

Any quadrilateral with the node n has two edges containing n; from this observation and assumption (7) above we can conclude

$\boldsymbol{2*q = 2*i + b}$.

Thus, $\boldsymbol{b}$ is even. This concludes the proof.

Prop 2. The boundary of any mesh has an even number of edges.

Proof- is similar to the one above. Let $\boldsymbol{b}$ be the number of boundary edges, let $\boldsymbol{i}$ be the number of internal edges, and $\boldsymbol{q}$ the total number of quadrilaterals. Then

$\boldsymbol{4*q = 2*i + b}$.

Thus, $\boldsymbol{b}$ is even. This concludes the proof.

Prop 3 The number of edges in the boundary of a mesh is greater than or equal to 4 if it is not empty.

Proof- by prop. 2 the number of edges is even. Suppose the boundary has two edges. By Prop., the nodes of the boundary are contained by an even number of of boundary edges. Hence the boundary, containing two edges must contain two nodes only. But, by assumption (3) above, the two edges would be the same; this is a contradiction. Hence the number of edges must be greater than two, but, being even, must be greater than or equal to four. This concludes the proof.

Def. 3 The dual of a mesh **M**, is the graph **M* = <Q, E, f>** where

Q is the set of quadrilaterals,

E is the set of internal edges of **M**,

And **f** is the function that maps each edge in E to the set of two unique quadrilaterals that contains it.

This definition has less structure than the STC [2].

When assumption (9) holds, dual of the graph is a 4-regular graph. In this case, the boundary of its quadrilaterals Q is empty, because every edge is shared by two quadrilaterals.

Existence of a perfect match

Def. 4 A mesh M is said to be connected if the graph Q* is connected.

Prop. 4 Let M be a connected mesh such that assumption (9) is satisfied, and C be a formal subset of a Q. The number of edges connecting C to Q*-C is greater than or equal to 4.

Proof- if C is a formal subset of Q, Q-C is not empty. The graph Q* is connected because M is connected by definition 4; therefore, the set of edges D between Q*-C and C cannot be empty.

Every edge of D is contained by the boundary of C, because only one element of C contains it. Every edge e in the boundary of C must be contained by exactly two quadrilaterals. Being a boundary edge of C, one of the quadrilaterals of C contains the edge e, and the other quadrilateral containing e must be in Q-C. (There is another quadrilateral because Q is connected.) Therefore the boundary of C is contained by D. By Prop. 3, this set must be greater than or equal than 4.

Def. 5 For a subset S of Q, $odd_{Q*}(S)$ is then number of connected components of Q*-S with and odd number of quadrilaterals.

Prop. 5 If a mesh M is such that assumption (9) is satisfied, and Q has an even number of quadrilaterals, Q* has a perfect matching.

Let S be a subset of Q. If S = ∅, the only connected component in Q*-S is Q*. Q has an even number of elements; hence $odd_{Q}(\emptyset) = 0$.

Let $\{C_i\}$ be the set of connected components with an odd number of quadrilaterals in Q*-S. The boundary of the the quadrilaterals of C_i is precisely the set of edges connecting C_i to S. By proposition 4, the number m_i of edges must be at least 4. Thus we have

$$4 \leq m_i$$

The total number of edges $\Sigma\, m_i$ connecting Q*-S to S must be less than or equal to the total number of quadrilaterals of S times four. Thus

$$4 * odd_{Q*}(S) \leq \Sigma\, m_i \leq 4 * |S|,$$
$$odd_{Q*}(S) \leq |S|.$$

This proves that Tutte's theorem can be applied.

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