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TECHNICAL REPORT

A One-Dimensional Model of the Navier-Stokes

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Abstract:

A one-dimensional nonlinear dynamical system is examined as a simplified model of the dynamics ensued by the Navier-Stokes equations. The model has a richer dynamical behaviour than the Burgers equation and shows several features similar to the ones that are associated with the three-dimensional Navier-Stokes. Although the spatial dimension is only one, there are still three velocity components and three "directions." The gradients along the transverse (virtual) directions are given by the product of the gradient along the real dimension and two arbitrary parameters, α and β, which can be either constant or variable. In general, differentiation with respect to the y- and z-axis is replaced by differentiation in the x-axis and multiplication by α and β, respectively. This model, for various values of the two parameters, is solved numerically with a pseudo-spectral method and the results are analyzed. The dynamics of the proposed model differs from the well studied dynamics of the Burgers equation. For example, in the case of variable coefficients, the shock formation which characterizes Burger-like solutions is not present in the proposed model.

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1 Introduction

The problem of incompressible viscous flows at high Reynolds numbers remains a fluid dynamical puzzle of great interest. If we restrict our attention to the cases where the presence of geometric boundaries can be neglected, there are essentially two difficulties which hinder our analysis. The first is the nonlinearity inherent in the Navier-Stokes and the second is the three spatial dimensions of the general problem. These two difficulties are coupled, which is demonstrated by the fact that the reduction of dimensionality results in a significantly different physical problem. For example, in the case of inviscid two-dimensional flows both energy and enstrophy are constants of motion and lead to equilibria in which most of the energy accumulates at the largest spatial scales; a familiar situation known as the Einstein-Bose condensation in the case of an ideal boson gas (Kraichnan & Montgomery 1980). On the other hand, in inviscid three-dimensional flows the enstrophy is not conserved – although the helicity is conserved – and most of the energy (of a truncated Euler system) in equilibrium resides at the high-wavenumber regime. Moreover, there is not vortex stretching parallel to the vorticity axis in two-dimensional flows whereas this mechanism is considered essential for the energy and enstrophy cascade in three dimensions. Hence, the change of dimensions results in significant differences from the fluid mechanics, as well as from the statistical mechanics, perspective.

Nevertheless, the study of dynamical systems that are characterized by similar nonlinearities with the Navier-Stokes is a reasonable way to gain information about them. Such an attempt was made early on by Jan M. Burgers (1939), who derived the following equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \,, \tag{1}$$

as a one dimensional archetype of the Navier-Stokes dynamics; here $u(x,t)$ is the one-dimensional velocity field and ν plays the role of the kinematic viscosity or, if we consider the quantities to be non-dimensional, the role of the inverse of the Reynolds number. However, it is well known that Cole (1951) and Hopf (1950), independently of each other, noted that the Burgers equation can be written as the linear heat equation by employing a nonlinear transformation (see, for example, Whitham (1974)). Therefore, the Burgers equation is integrable and not chaotic (i.e. it is not sensitive to initial conditions), as one would expect from a model of turbulence. Some problems related to Burgers equation are far from trivial; they include nonlinear wave phenomena in acoustics, plasma physics, and surface growth, and they deserve further study by themselves (see Gurbatov *et al.* (1997) and references therein).

A fundamental difference between the Burgers flow model and the actual turbulent flows is the lack of a vorticity vector. In order to illustrate this, let us write the incompressible Navier-Stokes in the so-called rotational form, i.e.

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{\omega}\times\boldsymbol{u} - \nabla\left(\frac{p}{\rho} + \frac{u^2}{2}\right) + \nu\nabla^2\boldsymbol{u} \,, \tag{2}$$

where p is the pressure field, ρ and ν are the density and kinematic viscosity of the fluid, respectively, $\boldsymbol{u}$ is the three-dimensional velocity field, and $\boldsymbol{\omega}$ is the vorticity. A mere inspection of equation (2) shows that the nonlinearity that is included in the Burgers equation is not the one that contains vorticity. The latter term is often called the Lamb vector and is absent from the Burgers model. This simple observation motivated the work presented herein. Our goal was to derive a one-dimensional model of the Navier-Stokes equations that included a term equivalent to the Lamb vector. In addition, we have chosen to impose a "continuity" equation in the model system, so that we can mimic, as much as it is possible, the system of equations that describe the turbulent motion of incompressible viscous (Newtonian) fluids in three-dimensions.

2 The Model System

The primary dynamical variables of our model system are four, namely, u, v, w, and p. The first three are envisioned as three velocity components whereas the last plays the role of the pressure; therefore, we will simply refer to the first three variables as the velocity variables. We will also envision that they correspond to three different directions which form an orthogonal triplet and take values along the x-axis of a Cartesian system in a Euclidean three-dimensional space. Thus, they will all be functions of one spatial dimension (x) and of time (t), and they will form a vector denoted by $\boldsymbol{u} = (u, v, w)$. In this context, we are able to define a vorticity vector that will be denoted by $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$ and whose three variables, which are functions of the velocity components, are given by

$$\omega_x = \alpha \frac{\partial w}{\partial x} - \beta \frac{\partial v}{\partial x} , \tag{3}$$

$$\omega_y = \beta \frac{\partial u}{\partial x} - \frac{\partial w}{\partial x} , \tag{4}$$

and

$$\omega_z = \frac{\partial v}{\partial x} - \alpha \frac{\partial u}{\partial x} . \tag{5}$$

In the above equations α and β are arbitrary parameters; they can be constants, deterministic functions of space and time, or random spatio-temporal functions with a given probability distribution. However, notice that when α and β are functions of x, they must be dependent variables if we want the vorticity to remain divergence free. In the present article we will focus on the case where α and β are constant along x, although they may vary as functions of time. Notice that the above definition is equivalent to the definition of the vorticity as the curl of the velocity, if we formally define a nabla operator (∇_*) as follows:

$$\nabla_* \equiv (1,\, \alpha,\, \beta)^T \, \frac{\partial}{\partial x} . \tag{6}$$

It follows that the equivalent of the Lamb vector $\boldsymbol{l}$ should be given by the following triplet

$$l_x = w\,\omega_y - v\,\omega_z , \tag{7}$$

$$l_y = u\,\omega_z - w\,\omega_x , \tag{8}$$

and

$$l_z = v\,\omega_x - u\,\omega_y . \tag{9}$$

Lastly, we define the equivalent of the Bernoulli energy function (Φ) by

$$\Phi = p + \frac{u^2 + v^2 + w^2}{2} . \tag{10}$$

Our model dynamical system consists of four equations for the four unknowns, i.e. u, v, w, p. In vectorial form the first three equations can be written as follows

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{l} - \nabla_* \cdot \Phi + \epsilon \nabla_*^2 \boldsymbol{u} + \boldsymbol{F} , \tag{11}$$

or, in terms of the individual components, as

$$\frac{\partial u}{\partial t} = -l_x - \frac{\partial \Phi}{\partial x} + \epsilon \left(1 + \alpha^2 + \beta^2\right) \frac{\partial^2 u}{\partial x^2} + F_x , \tag{12}$$

$$\frac{\partial v}{\partial t} = -l_y - \alpha \frac{\partial \Phi}{\partial x} + \epsilon \left(1 + \alpha^2 + \beta^2\right) \frac{\partial^2 v}{\partial x^2} + F_y , \tag{13}$$

$$\frac{\partial w}{\partial t} = -l_z - \beta \frac{\partial \Phi}{\partial x} + \epsilon \left(1 + \alpha^2 + \beta^2\right) \frac{\partial^2 w}{\partial x^2} + F_z , \tag{14}$$

where ϵ is the equivalent of the inverse of the Reynolds number in the nondimensional Navier-Stokes equations, and we have also added a possible external force, i.e. $(F_x, F_y, F_z)^T$. The fourth equation is succinctly expressed in vectorial form, i.e.

$$\nabla_* \cdot \boldsymbol{u} = 0 \,, \tag{15}$$

Our goal is to study the dynamics of equations (12) to (15). In §2, we present a number of analytical results that can be derived directly from the model equations. In §3, we use a standard pseudo-spectral method and obtain numerical solutions under various flow conditions. A comparison to Burger's and a fully three-dimensional flow for different α and β configurations is conducted to reveal the fundamental characteristics of the proposed model. In §4, we summarize this work and present our conclusions.

2.1 Analytical results of the model system

The space of three tuples of functions that depend on x and t is of course a vector space. The model system employs the following operator over the vector space of three tuples $\nabla_* = \left(\frac{\partial}{\partial x}, \alpha\frac{\partial}{\partial x}, \beta\frac{\partial}{\partial x}\right)^T$ as a substitute for the three dimensional nabla operator, $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)^T$. In the case where α and β are constant with respect to x, several vector identities that are familiar hold true. For example, given any vector $\mathbf{A}$, the divergence of its curl vanishes, i.e. $\nabla_* \cdot (\nabla_* \times \mathbf{A}) = 0$ is an identity. The newly defined nabla is clearly a linear operator over the three tuples vector space.

Utilizing this newly defined operator, it would be expected that the classical relationships of fluid dynamics are maintained for the case where α and β are only functions of time. The evolution of the vorticity field is obtained by formally taking the curl of equations (12) to (15).

$$\frac{\partial w_x}{\partial t} = -\left(\alpha\frac{\partial l_z}{\partial x} - \beta\frac{\partial l_y}{\partial x}\right) + \epsilon\left(1 + \alpha^2 + \beta^2\right)\frac{\partial^2 w_x}{\partial x^2} + \alpha\frac{\partial F_z}{\partial x} - \beta\frac{\partial F_y}{\partial x}\,, \tag{16}$$

$$\frac{\partial w_y}{\partial t} = -\left(\beta\frac{\partial l_x}{\partial x} - \frac{\partial l_z}{\partial x}\right) + \epsilon\left(1 + \alpha^2 + \beta^2\right)\frac{\partial^2 w_y}{\partial x^2} + \beta\frac{\partial F_x}{\partial x} - \frac{\partial F_z}{\partial x}\,, \tag{17}$$

$$\frac{\partial w_z}{\partial t} = -\left(\frac{\partial l_y}{\partial x} - \alpha\frac{\partial l_x}{\partial x}\right) + \epsilon\left(1 + \alpha^2 + \beta^2\right)\frac{\partial^2 w_z}{\partial x^2} + \frac{\partial F_y}{\partial x} - \alpha\frac{\partial F_x}{\partial x}\,, \tag{18}$$

$$\nabla_* \cdot \boldsymbol{\omega} = 0\,. \tag{19}$$

By taking the divergence of the proposed model equations (12)-(15), the following equations are obtained

$$\nabla_* \cdot \boldsymbol{l} = -{\nabla_*}^2\Phi \tag{20}$$

$$\frac{\partial l_x}{\partial x} + \alpha\frac{\partial l_y}{\partial x} + \beta\frac{\partial l_z}{\partial x} = -\left(\frac{\partial^2\Phi}{\partial x^2} + \alpha\frac{\partial^2\Phi}{\partial x^2} + \beta\frac{\partial^2\Phi}{\partial x^2}\right)\,. \tag{21}$$

2.2 Physical discussion of the model system

Let us consider a three dimensional turbulent fluid flow and let us imagine that we can collect information about the velocities and the pressure only along a virtual straight line which, without loss of generality, we take it to be our x-axis. In that case, the three velocities and the pressure can be measured, and the gradients of these quantities can be calculated along the x-axis. Let us now take the system of the Navier-Stokes equations together with the equation of mass continuity and examine what are the unknown quantities involved. We have six components of the rate of deformation tensor as unknowns, i.e. $\partial u/\partial y$, $\partial v/\partial y$, $\partial w/\partial y$, $\partial u/\partial z$, $\partial v/\partial z$, $\partial w/\partial z$; we have the two gradients of pressure as unknowns, i.e. $\partial p/\partial y$, $\partial p/\partial z$; and we have the six second derivatives of the viscous term as unknowns, i.e. $\partial^2 u/\partial y^2$, $\partial^2 v/\partial y^2$, $\partial^2 w/\partial y^2$, $\partial^2 u/\partial z^2$, $\partial^2 v/\partial z^2$, $\partial^2 w/\partial z^2$. This makes a total of 16 unknown quantities. In other words, if we wanted to compare the results of the laboratory experiments along that virtual line with the results of a one dimensional model, we would have to provide these 16 quantities as an input to the model. Such a task could make the problem more complicated than the original three dimensional problem for the whole domain!

Our proposed model attempts to simplify the problem by defining only two parameters, the α and β that we already mentioned. The implications of this is that we effectively impose the following system of constraints:

$$\alpha\frac{\partial u}{\partial x} \equiv \frac{\partial u}{\partial y}\,, \qquad \alpha\frac{\partial v}{\partial x} \equiv \frac{\partial v}{\partial y}\,, \qquad \alpha\frac{\partial w}{\partial x} \equiv \frac{\partial w}{\partial y}\,, \qquad \beta\frac{\partial u}{\partial x} \equiv \frac{\partial u}{\partial z}\,,$$

$$\beta\frac{\partial v}{\partial x} \equiv \frac{\partial v}{\partial z}\,, \qquad \beta\frac{\partial w}{\partial x} \equiv \frac{\partial w}{\partial z}\,, \qquad \alpha^2\frac{\partial^2 u}{\partial x^2} \equiv \frac{\partial^2 u}{\partial y^2}\,, \qquad \alpha^2\frac{\partial^2 v}{\partial x^2} \equiv \frac{\partial^2 v}{\partial y^2}\,,$$

$$\alpha^2\frac{\partial^2 w}{\partial x^2} \equiv \frac{\partial^2 w}{\partial y^2}\,, \qquad \beta^2\frac{\partial^2 u}{\partial x^2} \equiv \frac{\partial^2 u}{\partial z^2}\,, \qquad \beta^2\frac{\partial^2 v}{\partial x^2} \equiv \frac{\partial^2 v}{\partial z^2}\,, \qquad \beta^2\frac{\partial^2 w}{\partial x^2} \equiv \frac{\partial^2 w}{\partial z^2}\,,$$

$$\alpha\frac{\partial p}{\partial x} \equiv \frac{\partial p}{\partial y}\,, \qquad \beta\frac{\partial p}{\partial x} \equiv \frac{\partial p}{\partial z}\,.$$

The above equations imply that the ratio α/β is a measure of the anisotropy for the gradients along the y and z directions, whereas the ratio α^2/β^2 is a measure of the anisotropy for the dissipation terms. Thus, our model examines a small subspace of the space of the Navier-Stokes solutions but it does offer the opportunity to study its dynamics with more detail than the full system allows us to.

Notice that the viscous term of the model system introduces an amount of dissipation that is proportional to the square of the wavenumber and the (averaged) square of the arbitrary coeficients. Hence, the values of α and β should not be arbitrarily large, since such a choice would bias the model dynamics towards that of a purely dissipative system.

Equation (15) is not an expression of mass continuity. If the density of the fluid is constant, say ρ, and we imagine that ρv and ρw represent the flux of fluid mass along the y and z directions, respectively, then the continuity equation for our model is simply stated as

$$u + \alpha v + \beta w = 0\,. \tag{22}$$

Hence, if the sum of the velocities is not equal to zero one has to assume that there are sources and sinks of mass along the x-axis.

2.3 Fourier analysis of the model system

The interactions among scales of motion within the domain of the proposed model are given in the Fourier-space view by the sum of collections of triadic interactions among subsets of Fourier modes in the one-dimensional but multi-variable Fourier space. In the Fourier-spectral description, the velocity and pressure fields are expanded as an infinite discrete set of Fourier modes,

$$u(x,t) = \sum_{k_1} \hat{u}_{k_1(t)} e^{\hat{i} k_1 x},$$

$$v(x,t) = \sum_{k_2} \hat{v}_{k_2(t)} e^{\hat{\imath} k_2 x},$$

$$w(x,t) = \sum_{k_3} \hat{w}_{k_3(t)} e^{\hat{\imath} k_3 x},$$

$$p(x,t) = \sum_{k_4} \hat{p}_{k_4(t)} e^{\hat{\imath} k_4 x},$$

with harmonic basis functions. Each Fourier mode contains scale information through its wavenumber k_n, directional information through $e^{\hat{\imath} k_n x}$, and phase information through the leading coefficient $\vec{u}_{k_n}$ or p_{k_4} where n represents the associated variable (i.e. u, v, w, and p correspond to 1, 2, 3, and 4, respectively). In Fourier-spectral space, the non-linear terms of the proposed model become

$$\begin{aligned}\mathcal{F}(l_x) &= \alpha \sum_{k_{1,2}} \sum_{k_{2,1}} \hat{u}_{k_{1,2}} \hat{v}_{k_{2,1}} \hat{\imath} k_{1,2} e^{\hat{\imath}(k_{1,2}+k_{2,1})x} + \beta \sum_{k_{1,1}} \sum_{k_{3,1}} \hat{u}_{k_{1,1}} \hat{w}_{k_{3,1}} \hat{\imath} k_{1,1} e^{\hat{\imath}(k_{1,1}+k_{3,1})x} - \\ &\quad \sum_{k_{2,1}} \sum_{k_{2,2}} \hat{v}_{k_{2,1}} \hat{v}_{k_{2,2}} \hat{\imath} k_{2,2} e^{\hat{\imath}(k_{2,1}+k_{2,2})x} - \sum_{k_{3,1}} \sum_{k_{3,2}} \hat{w}_{k_{3,1}} \hat{w}_{k_{3,2}} \hat{\imath} k_{3,2} e^{\hat{\imath}(k_{3,1}+k_{3,2})x}\end{aligned}$$

$$\begin{aligned}\mathcal{F}\left(\frac{\partial \Phi}{\partial x}\right) &= \sum_{k_4} \hat{p}_{k_4} e^{\hat{\imath} k_4 x} + \sum_{k_{1,1}} \sum_{k_{1,2}} \hat{u}_{k_{1,1}} \hat{u}_{k_{1,2}} \hat{\imath} k_{1,2} e^{\hat{\imath}(k_{1,1}+k_{1,2})x} + \\ &\quad \sum_{k_{2,1}} \sum_{k_{2,2}} \hat{v}_{k_{2,1}} \hat{v}_{k_{2,2}} \hat{\imath} k_{2,2} e^{\hat{\imath}(k_{2,1}+k_{2,2})x} + \sum_{k_{3,1}} \sum_{k_{3,2}} \hat{w}_{k_{3,1}} \hat{w}_{k_{3,2}} \hat{\imath} k_{3,2} e^{\hat{\imath}(k_{3,1}+k_{3,2})x}\end{aligned}$$

where the extra subscripts are used to keep track of separate summation indices. Even though the results are only given for the u-component, the results are the same in the transverse variables due to symmetry.

Notice that the last two (non-linear) terms in the above equations are eliminated when $-l_x - \frac{\partial \Phi}{\partial x}$ is evaluated. This is a result of the reduction in dimensionality of the Navier-Stokes equations. Hence, the proposed model is limited in so far as it does not model the dual non-linear triadic interactions within the transverse variables with no dependence on the other velocity variables. This means there are no $r = -(k_{2,1} + k_{2,2})$ or $r = -(k_{3,1} + k_{3,2})$ interactions for the x-direction, where, by orthogonality, r is the set of modes that could be excited by the given interactions (i.e. $\langle e^{\hat{\imath} r x}, \sum_{k_{2,1}} \sum_{k_{2,2}} e^{\hat{\imath}(k_{2,1}+k_{2,2})x)} \rangle = 0$). Note that the evolution of the negative wavenumbers can be obtained from the positive set from $u(-k_1) = u(+k_1)^*$, where the asterisk implies a complex conjugate, thus only half of the wavenumber space need be explicity analyzed. However, the proposed model does exhibit energy transfer between different variables as has been observed empirically; however, the nonlinear behavior is significantly damped relative to similar Burger's flow conditions as discussed in §3.

3 Results from the model

We will give now a brief description of the numerical method, the study cases of the numerical simulations, and their results. The numerical method is a standard pseudo-spectral method similar to the one that is used to solve the Navier-Stokes equations in the rotational form subject to the incompressibility constraint. Since we are interested in the fundamentals of the dynamics we will restrict ourselves to the case of periodic boundary conditions.

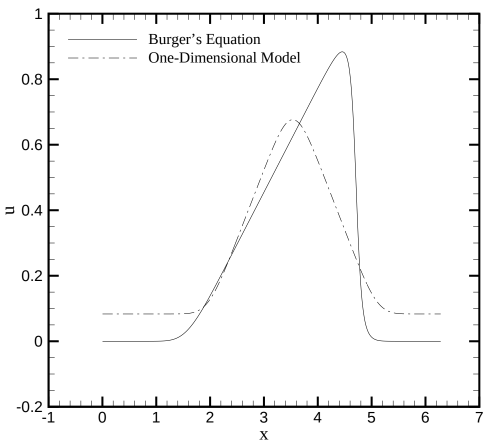

Figure 1: Solutions to the viscid Burgers' equation and the proposed model at time T_b where the inviscid case would form a shock.

3.1 Numerical Methodology

For the purposes of this paper, we are interested in the numerical approximation of equations (12)-(15) subject to periodic boundary conditions and parameters $\alpha = \alpha(t) \in \mathbb{R}$ and $\beta = \beta(t) \in \mathbb{R}$ (i.e. only functions of time, not space). As is commonly done in the numerical study of homogeneous turbulence, we have chosen to use a Fourier pseudo-spectral method in space with a 3/2 de-aliasing rule ([Canuto *et al.* 1988]). For all simulations, the parameter N denotes the number of evenly-spaced points per variable used on the interval $(0, 2\pi]$ (or, correspondingly, the number of Fourier modes employed for representing spatial variation). Since the new divergence operator mathematically mimics the traditional operator (see section 2.1), the splitting scheme presented in [Karniadakis, Israeli & Orszag] (1991) and [Karniadakis & Sherwin] (1999) was used to accurately integrate in time equations (12)-(14) while maintaining the "divergence-free condition" given by equation (15). All simulations presented herein were accomplished with second-order time integration using a stated Δt.

3.2 Example 1: A comparison to the viscid Burgers' equation

For the inviscid Burgers' equation, it is well known that, if the first derivative of the initial condition, $u_x(x, 0)$ is negative at any point in the domain, the characteristics of the differential equation will cross resulting in a shock formation in the solution. This example will employ an initial condition that would replicate this condition for the inviscid case in order to compare the shock formation characteristic of the viscid Burgers' equation and the proposed one-dimensional model. The initial condition is a piecewise linear function of the form $u(0 < x < \frac{\pi}{2}, 0) = u(\frac{3\pi}{2} < x \leq 2\pi, 0) = 0$, $u(\frac{\pi}{2} \leq x \leq \pi, 0) = \left(\frac{2}{\pi}\right) x - 1$, and $u(\pi < x \leq \frac{3\pi}{2}, 0) = \left(\frac{-2}{\pi}\right) x + 3$. For the inviscid case, a shock will appear at time $T_b = \frac{\pi}{2}$. The solution $u(x, T_b)$ for the viscid Burger's equation and the proposed model with $N = 2048$, $\nu = 0.005$, $\Delta t = 0.0002$, $\alpha = 1$, and $\beta = 1$ is shown graphically in figure 1. Effectively, the proposed model does not propagate information to the higher wavemodes of the solution, in effect, it dampens them resulting in a smoothing effect that is seen in figure 1. The proposed model will not form shocks for smooth initial conditions.

3.3 Example 2: Free decay with constant α and β

The following is an analysis of the time-history progression of a one-dimensional line of the velocity profile from HyperBox, a fully three-dimensional spectral code for the solution of the incompressible Navier-Stokes equations in a periodic box (Marmanis 1999). This will allow for the characteristics of the proposed one-dimensional model to be compared to that of a fully three-dimensional DNS simulation for a long time integration period.

HyperBox was run for a total of twenty convective time units under the condition of natural decay starting with random initial conditions of a prescribed isotropic energy spectra and energy level. A constant timstep $\Delta t = 0.0002$, viscosity $\nu = 0.02$, and resolution $N = 128^3$ were used. At $t_0 = 2$, a line from the HyperBox solution along $[x, \pi, \pi]$ including the u, v, and w data was extracted. Similar extractions from the HyperBox solution were performed every 0.1 convective units until the solution was stopped at $t = 20$ in order to consistenly compare the HyperBox and one-dimensional model solutions. The line at $t_0 = 2$ was then imported into the one-dimensional solver using the appropriate Fourier methods and allowed to run to $t = 20$ using the same conditions ($\Delta t = 0.0002, \nu = 0.02, N = 128$) as used in HyperBox. This particular one-dimensional simulation was performed using $\alpha = 1$ and $\beta = 1$. Therefore, the decay is isotropic in both the gradient and diffusion terms.

Let us define the error between the HyperBox and proposed model solutions for the u-component of the velocity as $e_u = u_{HyperBox} - u_{model}$. The error between the other components are defined similarly. Hence, discrete the L^2 error and L^∞ error (for u) are

$$\|e_u\|_2 = \sqrt{\frac{1}{N}\sum_{k=1}^{N} {e_{u_k}}^2} \ , \tag{25a}$$

$$\|e_u\|_\infty = \max_k |e_{u_k}| \ . \tag{25b}$$

The variation of the discrete L^2 and L^∞ norms as a function of time are shown in figures 2 and 3, respectively. Naturally, it was not expected that a one-dimensional model would be able to accurately predict the decay of u, v, or w, independently. The limitations imposed by the reduction in dimensionlity are evident as shown in §2 as well as the HyperBox solution along $[x, \pi, \pi]$ being influenced through the transmittal of energy to and from its surroundings in the three-dimensional lattice unlike the one-dimensional model, which is not externally modified. However, the bulk dynamics of the flow as given by $\langle u^2 \rangle$ $(= \frac{1}{N}\sum_k \boldsymbol{u}_k \cdot \boldsymbol{u}_k)$ are followed as shown in figure 4. This characteristic of the proposed model is enhanced further for time-varying α and β as shown in the next example.

3.4 Example 3: Free decay with time-varying α and β

In this example, the same simulation parameters were used as in the previous example except that both α and β were modeled as functions of time. In order to more accurately model the $\langle u^2 \rangle$ decay of the HyperBox solution, the proposed model parameters, α and β, were allowed to vary in time as modified distribution functions,

$$\alpha(t) = \beta(t) = 7\left(\frac{1}{2\left(1 + e^{-(t-t_0)}\right)} - \frac{1}{4}\right) \ . \tag{26}$$

Since the previous example demonstrated that the decay of the HyperBox solution was not modeled well during the initial stages of the decay where the convective terms were dominant with an α and

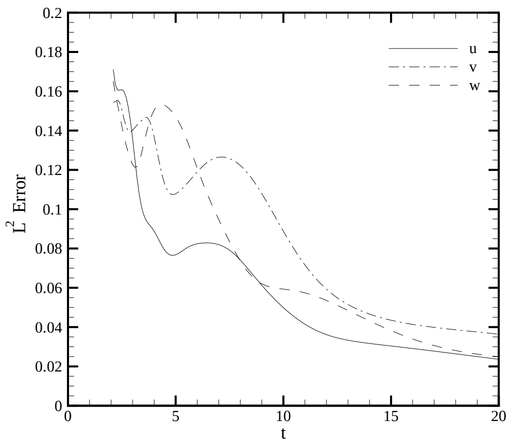

Figure 2: Discrete L^2 error in the solutions of HyperBox and the one-dimensional model for constant α and β.

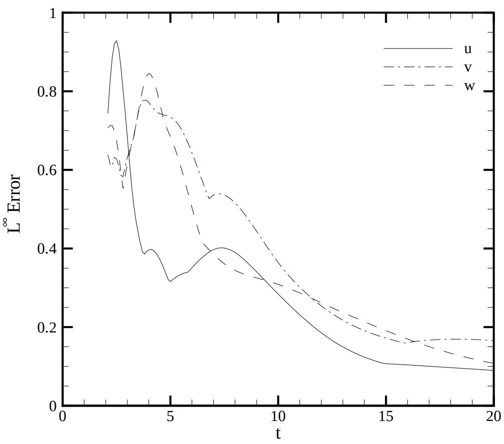

Figure 3: Discrete L^∞ error in the solutions of HyperBox and the one-dimensional model for constant α and β.

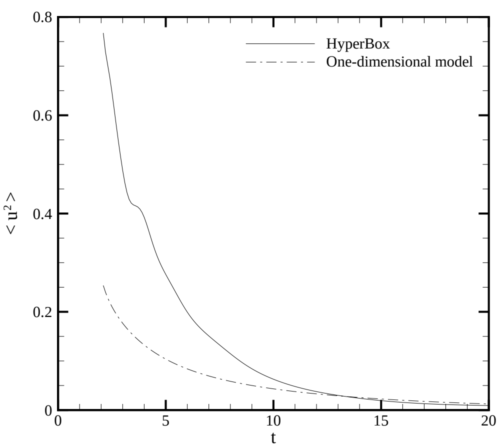

Figure 4: The $\langle u^2 \rangle$ decay for the HyperBox and one-dimensional model solutions with constant α and β.

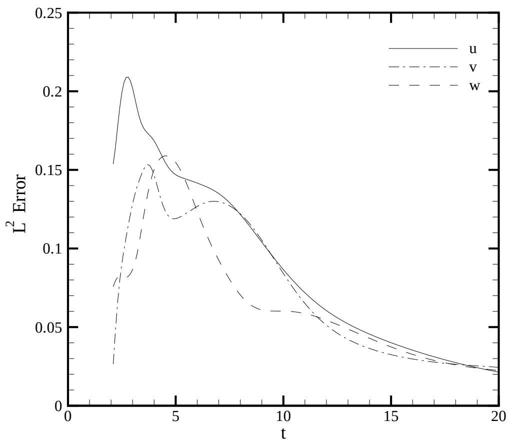

Figure 5: Discrete L^2 error in the solutions of HyperBox and the one-dimensional model for time-varying α and β.

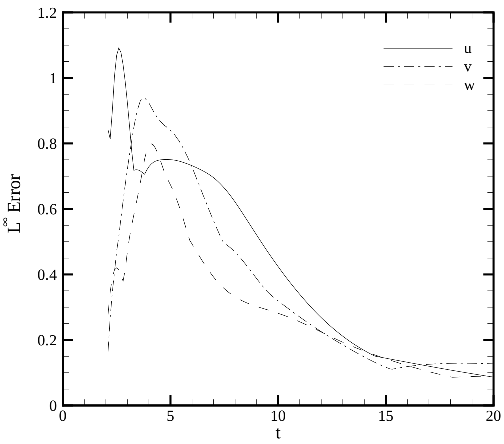

Figure 6: Discrete L^∞ error in the solutions of HyperBox and the one-dimensional model for time-varying α and β.

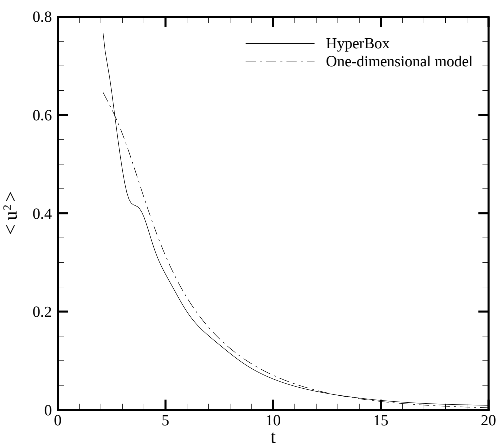

Figure 7: The $\langle u^2 \rangle$ decay of the HyperBox and one-dimensional model solutions with time-varying α and β.

β combination that further amplifies the effect of the viscous terms, the α and β parameters as given by equation 26 allowed for the proper transferal of the influence of the convective and viscous terms during the initial and final stages of decay, respectively. Effectively, viscous diffusion is damped in the initial stages of the decay where the flow dynamics are principally governed by the convective terms and the viscous diffusion is made dominant towards the end of the simulation where viscous forces are most influential. The variation of the discrete L^2 and L^∞ norms are shown in figures 5 and 6, respectively. Notice that the time-variation of the error in the solution is mitigated by the introduction of time-varying α and β. Similarly, figure 7 shows a pronounced increase in accuracy of the dynamics of the flow for the proposed model.

4 Summary

A one-dimensional non-linear model that attempts to capture the dynamics of the Navier-Stokes equations was considered. An orthogonal triplet composed of the time-varying velocity variables along a one-dimensional axis was formed. Due to the reduction of dimensionality, derivatives in the transverse directions were considered to be related to the derivatives along the primary axis by arbitrary parameters α and β. This simplification was shown to retain many of the non-linear behaviors commonly associated with the Navier-Stokes equations. When α and β were taken as time-varying but spatially constant variables, the $\langle u^2 \rangle$ decay of homogenous turbulence decay was found to be well approximated by the aforementioned model. The particular form of the time-varying parameters was chosen so as to either enhance inertial effects when the convective terms dominated the transport equations or amplify the viscous effects during the latter stages of decay through the system parameters α and β.

However, extension to wall-bounded or non-homogenous flow fields appears intractable without determining the relationship between various derivatives of the flow field *a priori*. The choice of the x-direction as the primary variable is difficult to justify. For example, in a fully-developed laminar channel flow, the derivative $\alpha\frac{\partial u}{\partial x}$ would be identically zero; however, the wall-normal derivative $\frac{\partial u}{\partial y} = \alpha\frac{\partial u}{\partial x}$ would not be zero (specifically, the streamwise velocity would have a parabolic profile). Hence, using this model for global dynamics is unproductive at best. Though, for local dynamics where isotropy can be assumed, this model may hold merit.

This approach attempts to model the *equations* of motion rather than the underlying *physics* of turbulence by imposing a set of constraints on the derivatives in the transverse directions with respect to the primary dimension without any physical justification. Rearranging or deleting terms of the governing equations without sufficient physical justification often leads to interesting dynamical systems equations that have little predictive power in the space of fluid dynamics solutions.

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