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TECHNICAL REPORT

ICE Algorithm and the Davis Advection Scheme

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January 23, 2006

Abstract:

As an intermediate step towards understanding Kashiwas full ICE algorithm for the Euler equations, we start by implementing and analyzing Davis scheme [Dav87] upon which ICE is based [Kas00]. We describe the algorithm and the results for a linear advection equation, a non-linear advection equation, and for the Euler equations (Sods shocktube problem).

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As an intermediate step towards understanding Kashiwa's full ICE algorithm for the Euler equations, we start by implementing and analyzing Davis' scheme [Dav87] upon which ICE is based [Kas00]. We describe the algorithm and the results for a linear advection equation, a non-linear advection equation, and for the Euler equations (Sod's shocktube problem).

Key words. ICE, TVD schemes, positivity preserving, flux limiter, sonic flux.

1 Introduction

We consider the system of conservation laws

$$U_t + (F(U))_x = 0 \tag{1}$$

where U and F are vectors of length m. The Davis scheme is a modified MacCormack Scheme [Lan98, p. 360]. Instead of having plane second-order spatial accuracy everywhere at the cost of oscillations near shock fronts, Davis adds to the numerical flux a dissipative term controlled by a slope limiter $\phi(r)$, where r is the approximate local ratio of neighboring solution slopes. In smooth regions, we use $\phi = 1$ to obtain the second-order MacCormack scheme. When r is small, we expect a discontinuity and reduce to a first-order upwind method, which is less accurate, but preserves monotonicity (or quite equivalently, preserves positivity) of the solution profile;

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this is obtained by $\phi = 0$. In general, changing ϕ can be utilized to make the transition from a smooth region to a shock viscinity.

The paper [Dav87] describes an algorithm for time-integration of conservation laws. The algorithm applies to scalar equations and to systems of conservation laws in virtually the same way (except some minor treatment of the local Courant number used by the scheme). We will describe and analyze the algorithm for systems. For simplicity we restrict ourselves to one space dimension. A future report will discuss extensions to two and three space dimensions.

The report is organized as follows. The algorithm of the Davis Scheme is described in §2. We analyze the numerical flux and monotonicity properties of the scheme in §3. Numerical results for a linear advection equation are presented in §4. We compare two second-order base methods (MacCormack and Richmyer) inside the Davis scheme in §4.1. We compare two limiters (Davis and van-Leer) in §4.2. Results for a non-linear advection equation (Burgers') are discussed in §5. Sod's shocktube problem (compressible Euler equations) is considered in §6. We summarize our findings in §7.

2 Davis' Scheme: Algorithm

In what follows we ignore boundary conditions. We assume a periodic or infinite domain. The system (1) is discretized on a uniform grid with meshsize Δx in space, and a uniform grid with timestep Δt in time. Nodes are defined at grid cell centers. The solution at time n and point j is denoted by U_j^n. When the superscript is omitted, we refer to time t_n (for instance, U_j refers to U_j^n). "Points" j refer to spatial locations $j\Delta x$; similarly n refers to time $t_n = n\Delta t$. Half-indices ($j+\frac{1}{2}$ and $n+\frac{1}{2}$) refer to mid-grid-cell locations and mid-times, respectively.

Davis' algorithm is an explicit time-integrator. From previously computed values $\{U_j^n\}_j$, we compute U_j^{n+1} for all j using the following steps [Dav87, p. 11]. Some of Davis' notation have been modified for clarity.

1. Compute the solution differences

$$\Delta U_{j+\frac{1}{2}} \leftarrow U_{j+1} - U_j, \qquad \forall j. \tag{2}$$

2. Prepare the dissipation terms. For all j, compute

$$D_{j+\frac{1}{2}} \leftarrow 0.5C(\nu_j)\left(2 - \phi\left(r_j^+\right) - \phi\left(r_{j+1}^-\right)\right)\Delta U_{j+\frac{1}{2}}, \tag{3}$$

where

$$C(\nu) := \min\{\nu(1-\nu), 0.25\}, \qquad \nu := \rho(A(U_j))\lambda, \qquad \lambda := \frac{\Delta t}{\Delta x}, \tag{4}$$

$A(U)$ is the Jacobian matrix of F, and $\rho(A(U))$ is its spectral radius. Note that explcitly computing $A(U)$ is not always necessary. In many cases, the eigenvalues of $A(U)$ are known (e.g., for the Euler equations). Finally, the slope ratios

$$r_j^+ := \frac{\left(\Delta U_{j-\frac{1}{2}}, \Delta U_{j+\frac{1}{2}}\right)}{\left(\Delta U_{j+\frac{1}{2}}, \Delta U_{j+\frac{1}{2}}\right)}, \quad r_j^- := \frac{\left(\Delta U_{j+\frac{1}{2}}, \Delta U_{j-\frac{1}{2}}\right)}{\left(\Delta U_{j-\frac{1}{2}}, \Delta U_{j-\frac{1}{2}}\right)}, \tag{5}$$

control the amount of artificial viscosity generated by D through a limiter of choice. The notation $(\cdot,\cdot)$ represents an inner product on $\mathbb{R}^m$, namely,

$$(A, B) := \sum_{k=1}^{m} A_{(k)} B_{(k)}, \qquad \forall A, B \in \mathbb{R}^m \tag{6}$$

($A_{(k)}$ refers to the kth component of the vector A). We use Davis' limiter

$$\phi(r) := \begin{cases} \min\{1, 2r\}, & \text{if } r > 0, \\ 0, & \text{if } r \le 0 \end{cases} \tag{7}$$

as a default. Another choice of interest is the van Leer limiter,

$$\phi(r) := \frac{r + |r|}{1 + |r|}. \tag{8}$$

3. Compute a provisional solution using the MacCormack method (or in general, a base second-order method such as Lax-Wendroff, Richtmyer, etc.). This is a predictor-corrector method [Lan98, p.356], that first computes a mid-time solution

$$U_j^{(1)} \leftarrow U_j - \lambda \left[F\left(U_j\right) - F\left(U_{j-1}\right) \right] \tag{9}$$

for all j, and then uses it to find its final result,

$$U_j^{(2)} \leftarrow 0.5 \left\{ U_j + U_j^{(1)} - \lambda \left[F\left(U_{j+1}^{(1)}\right) - F\left(U_j^{(1)}\right) \right] \right\} \tag{10}$$

for all j.

4. Add the dissipative terms to the provisional solution to get the final result of Davis' scheme,

$$U_j^{n+1} \leftarrow U_j^{(2)} + \left(D_{j+\frac{1}{2}} - D_{j-\frac{1}{2}}\right). \tag{11}$$

Notice that U_j^{n+1} depends on a five-point stencil at time n, namely, $U_{j-2}, \ldots, U_{j+2}$. Only three points are used in the basic second-order scheme in the next step, but $r_j^{\pm}$ involve gradients that depends on five points.

3 Analysis of the Scheme

To analyze the effect of the limiter, consider first the scalar linear advection equation

$$u_t + au_x = 0, \qquad 0 \le x \le 1, \quad t > 0. \tag{12}$$

we write Davis' scheme in conservative form,

$$U_j^{n+1} = U_j + \frac{\Delta t}{\Delta x}\left(F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}\right), \tag{13}$$

where

$$F_{j+\frac{1}{2}} := aU_j + \frac{a(1-|\nu|)}{2}\left(\phi\left(r_j^+\right) + \phi\left(r_{j+1}^-\right) - 1\right). \tag{14}$$

The first term corresponds to first order upwind scheme. The second term becomes the Lax-Wendroff correction term that makes the entire scheme second order, if $\phi \equiv 1$.

4 Results for Linear Advection

To obtain a first idea on how the Davis scheme works, we reproduce his result for the scalar linear advection equation (12). Here $a = 1$, $U = u$ and $F(u) = u$. The initial data is the square wave

$$u(x) := \begin{cases} 0, & \text{if } 0 \le x < 0.1, \\ 1, & \text{if } 0.1 < x < 0.3, \\ 0, & \text{if } 0.3 < x \le 1. \end{cases} \tag{15}$$

We discretize in space using $N = 100$ points and $\Delta x = 0.01$, and advance in time, but stop before any boundary effects enter.

4.1 Choice of Base Scheme

The “base scheme” for Davis’ algorithm is the second-order scheme in step 3. We compare the default choice of MacCormack’s scheme with the Richtmyer scheme [Lan98, p. 360], that computes a provisional solution from U^n as follows.

$$U_{j+\frac{1}{2}}^{n+\frac{1}{2}} \leftarrow \frac{1}{2}\left(U_{j+1}+U_j\right)-\frac{\lambda}{2}\left(F\left(U_{j+1}\right)-F\left(U_j\right)\right), \quad \forall j \tag{16}$$

$$U_j^{n+1} \leftarrow U_i-\lambda\left(F\left(U_{j+\frac{1}{2}}^{n+\frac{1}{2}}\right)-F\left(U_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right)\right), \quad \forall j. \tag{17}$$

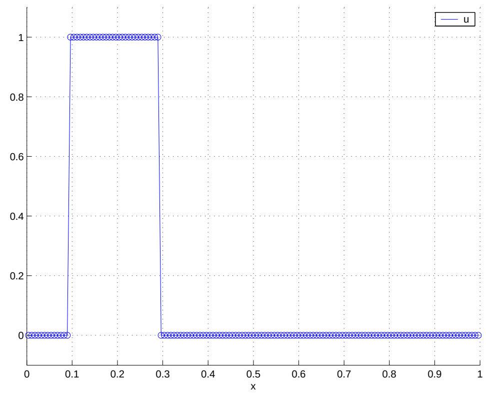
We first test a MacCormack base scheme with Davis’ limiter (7). Figs. 3-4 show the solution profile at different times, for a quite large CFL = 0.7 (CFL = 0.9 reported in Davis to be stable with the Lax-Wendroff scheme, appears undestable in our experiments with the MacCormack scheme. The largest stable CFL seems to be ≈ 0.75). The figures look identical to the result [Dav87, Fig. 2c], hence we can assume the implementation is correct and the scheme works for scalar linear advection. Note that the solution profile does not have oscillations; the scheme automatically switches to a first-order method, and some smearing occurs near the shock locations, instead of the “wiggles” normally produced by a stand-alone MacCormack scheme, presented in Figs. 1–2.

Next, we compare the MacCormack base scheme with a Richtmyer base scheme (16)–(17). We use the same limiter (Davis); see Figs. 5–6. The results are indistinguishable from the MacCormack results, hence we can use either one. For consistency with Davis’ paper, we will use MacCormack scheme from here on.

4.2 Choice of Limiter

We now compare two limiters with the MacCormack base scheme inside Davis’ algorithm: the Davis limiter (7), and the van Leer limiter (8). The first one was already demonstrated in Figs. 3-4. The same scheme with van Leer’s limiter is depicted in Figs. 7-8.

Notice that the solution has new local extrema, noticably near the right shock front, which the Davis limiter results did not exhibit. This is because the Davis scheme is positivity preserving only for the first limiter. The rest of this section is devoted to finding conditions on the limiter for which the scheme is positivity preserving.



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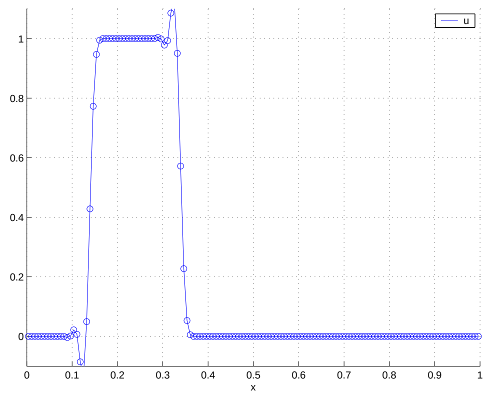

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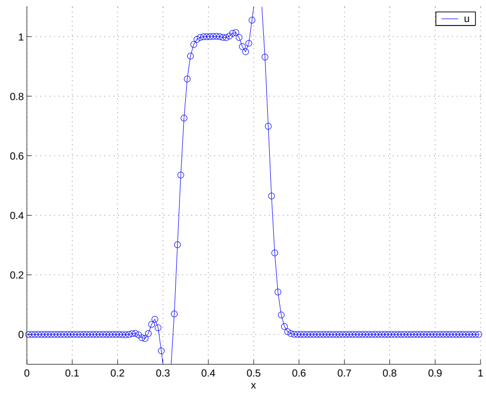
Figure 1: Results of the plain MacCormack scheme for linear scalar advection (12). Here CFL $= 0.7$, so $\Delta t = 0.7 \Delta x = 0.07$. (a) Initial data at $t = 0$. (b) Solution after 10 timesteps.



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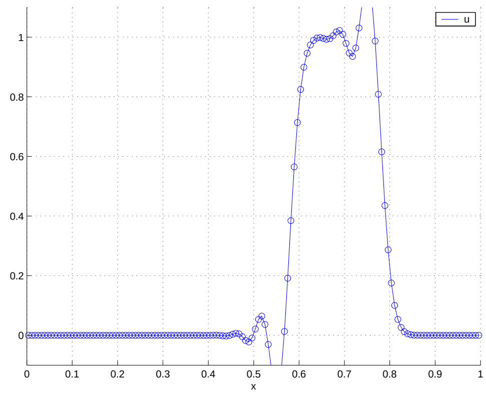

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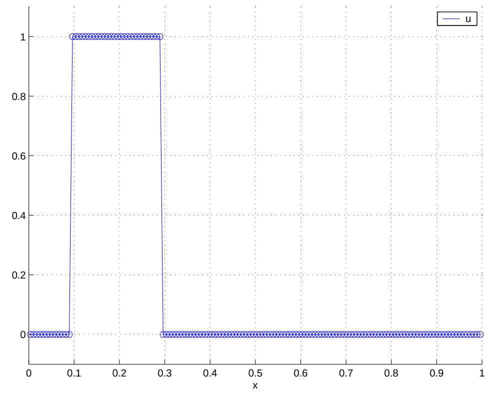
Figure 2: Results of the plain MacCormack scheme for linear scalar advection (12). Here CFL $= 0.7$, so $\Delta t = 0.7\Delta x = 0.07$ (continued from Fig. 1). (c) Solution after 50 timesteps. (d) Solution after 100 timesteps.



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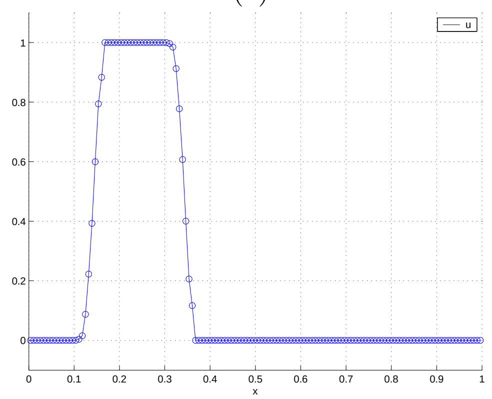

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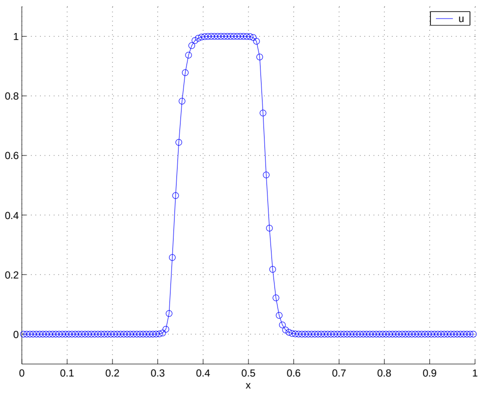
Figure 3: Results of the Davis scheme for linear scalar advection (12), with the MacCormack scheme and Davis limiter. Here CFL $= 0.7$, so $\Delta t = 0.7\Delta x = 0.07$. (a) Initial data at $t = 0$. (b) Solution after 10 timesteps.



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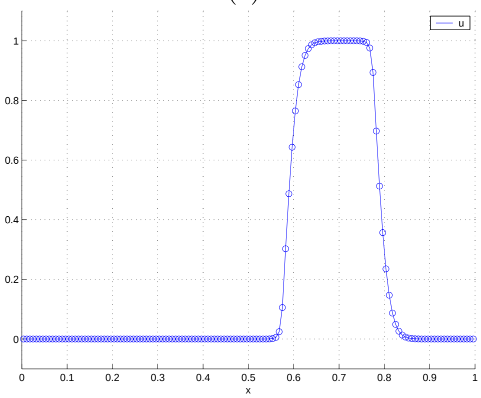

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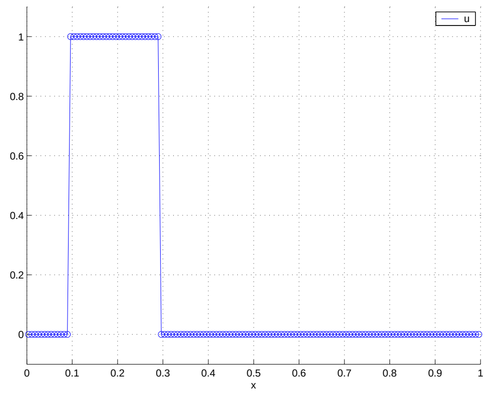
Figure 4: Results of the Davis scheme for linear scalar advection (12), with the MacCormack scheme and Davis limiter. Here CFL $= 0.7$, so $\Delta t = 0.7\Delta x = 0.07$ (continued from Fig. 1). (c) Solution after 50 timesteps. (d) Solution after 100 timesteps.



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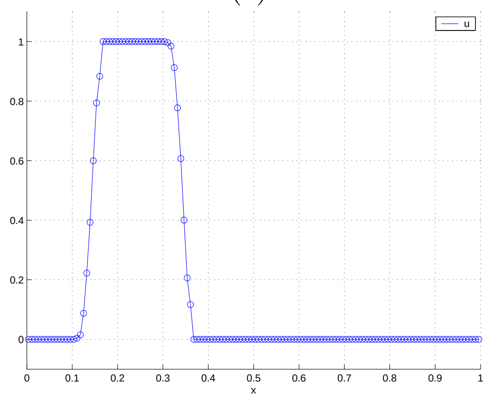

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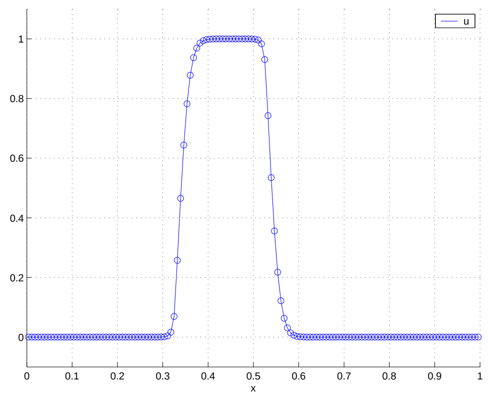
Figure 5: Results of the Davis scheme for linear scalar advection (12), with the Richtmyer scheme and Davis limiter. Here CFL $= 0.7$, so $\Delta t = 0.7 \Delta x = 0.07$. (a) Initial data at $t = 0$. (b) Solution after 10 timesteps.



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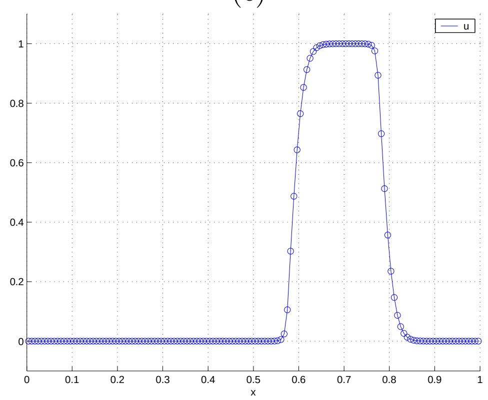

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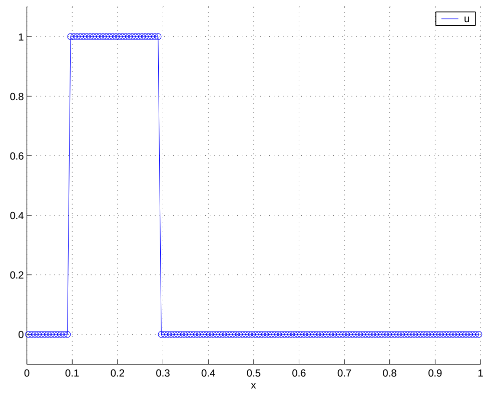
Figure 6: Results of the Davis scheme for linear scalar advection (12), with the Richtmyer scheme and Davis limiter. Here CFL $= 0.7$, so $\Delta t = 0.7\Delta x = 0.07$ (continued from Fig. 1). (c) Solution after 50 timesteps. (d) Solution after 100 timesteps.



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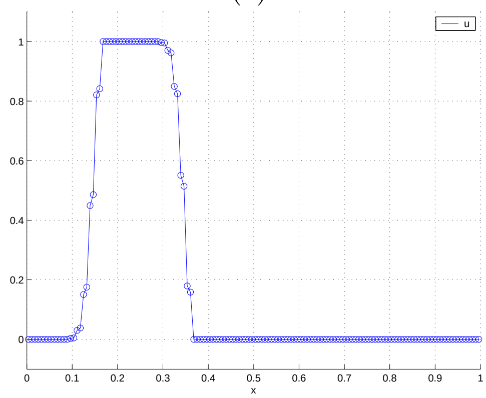

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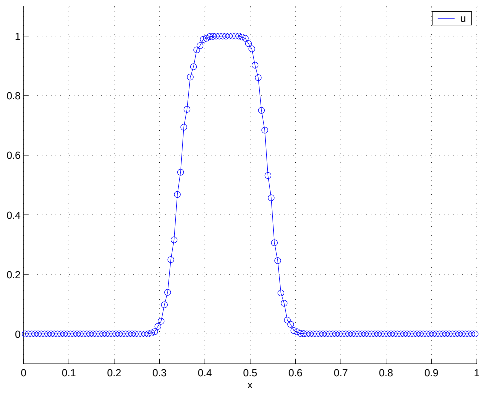
Figure 7: Results of the Davis scheme for linear scalar advection (12), with the MacCormack scheme and van Leer limiter. Here CFL $= 0.7$, so $\Delta t = 0.7\Delta x = 0.07$. (a) Initial data at $t = 0$. (b) Solution after 10 timesteps.



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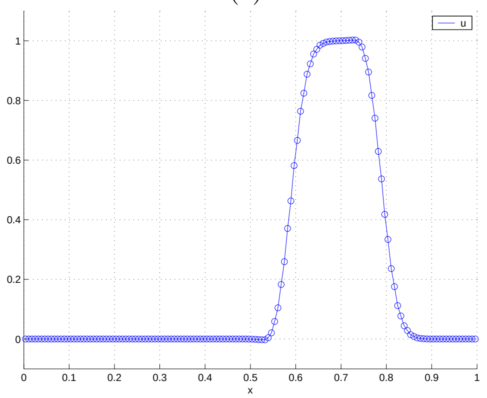

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Figure 8: Results of the Davis scheme for linear scalar advection (12), with the MacCormack scheme and van Leer limiter. Here CFL $= 0.7$, so $\Delta t = 0.7\Delta x = 0.07$ (continued from Fig. 1). (c) Solution after 50 timesteps. (d) Solution after 100 timesteps.

5 Results for Burgers' Equation

We now test a non-linear scalar advection example, the Burger's equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \tag{18}$$

with the same initial data (15) as for linear advection. The results of the Davis scheme (MacCormack base, Davis limiter) are depicted in Figs. 9-10. Again, they are in line with [Dav87, Fig. 3c] and there are even no "gliches" as in the upwind-dependent variant of Davis' scheme [Dav87, Fig. 3b].

Corresponding experiments with the van Leer limiter show spikes near the right shock front (corresponding to large r, see §4.2), which however are of bounded magnitude. See Figs. 11-12.

Note, however, that even for Davis' limiter, the scheme generates a new extremum near $x = 0.1$ (on the left of the rarefraction shock). This is a matter of concern, because the scheme can probably be shown to be positive for Burgers' equation, just like for the linear advection case, and this should not happen.

6 Results for Sod's Shocktube Problem

Compressible Euler equations, see Figs. 13-14.

7 Conclusions

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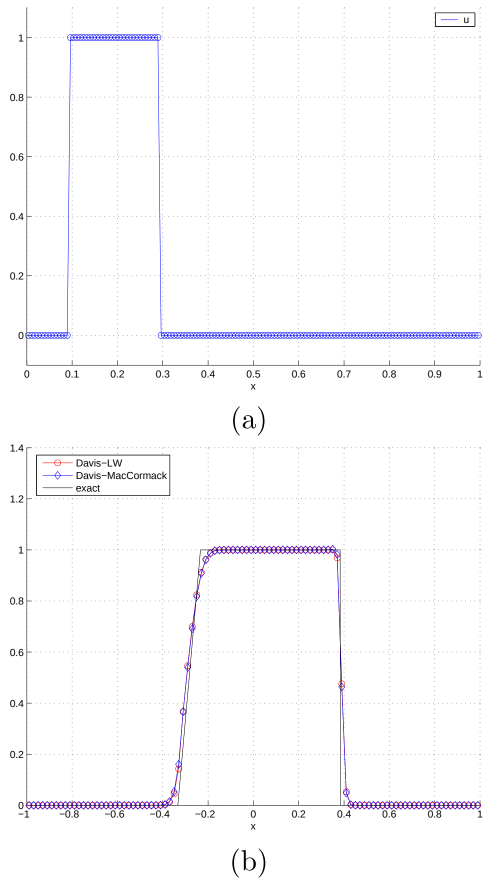
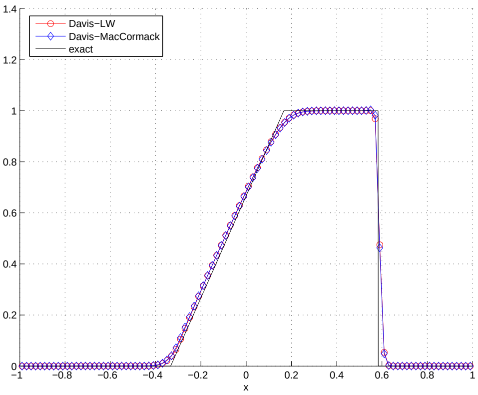


Figure 9: Results of the Davis scheme for for Burgers' equation (18) with the MacCormack scheme and Davis limiter. Here CFL $= 0.5$, so $\Delta t = 0.5 \Delta x = 0.05$. (a) Initial data at $t = 0$. (b) Solution after 10 timesteps.



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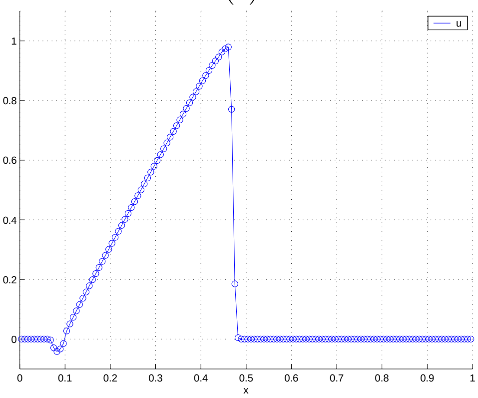

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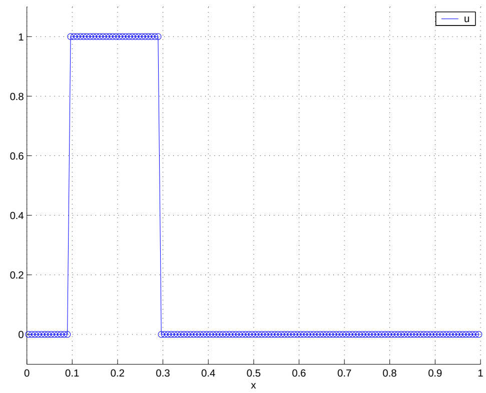
Figure 10: Results of the Davis scheme for Burgers' equation (18) with the MacCormack scheme and Davis limiter. Here CFL $= 0.5$, so $\Delta t = 0.5\Delta x = 0.05$ (continued from Fig. 9). (c) Solution after 50 timesteps. (d) Solution after 100 timesteps.



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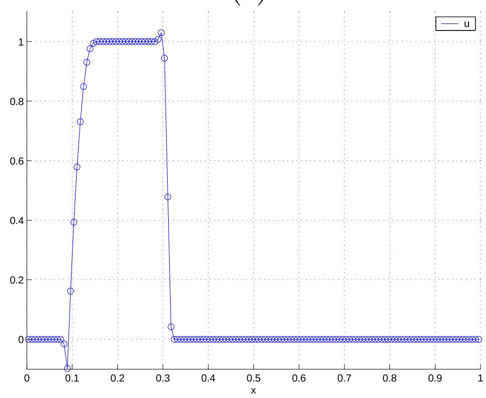

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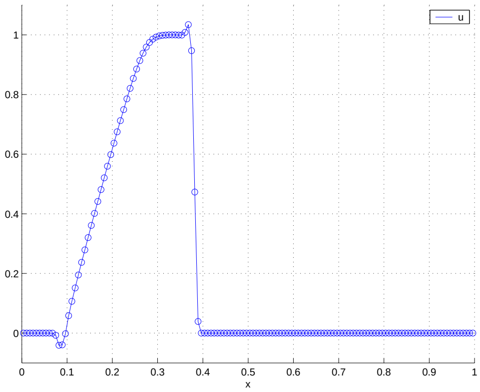
Figure 11: Results of the Davis scheme for for Burgers' equation (18) with the MacCormack scheme and van Leer limiter. Here CFL = 0.5, so $\Delta t = 0.5\Delta x = 0.05$. (a) Initial data at $t = 0$. (b) Solution after 10 timesteps.



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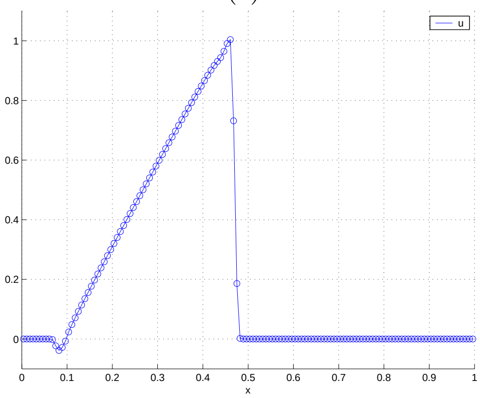

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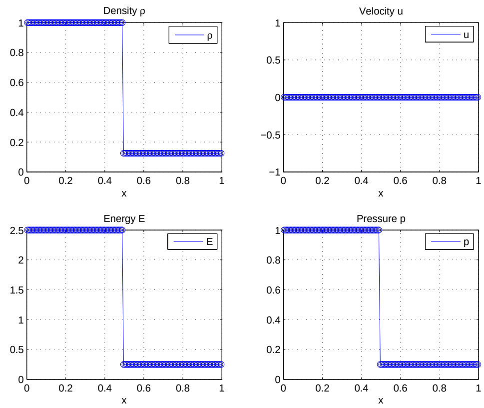
Figure 12: Results of the Davis scheme for Burgers' equation (18) with the MacCormack scheme and van Leer limiter. Here CFL $= 0.5$, so $\Delta t = 0.5\Delta x = 0.05$ (continued from Fig. 11). (c) Solution after 50 timesteps. (d) Solution after 100 timesteps.



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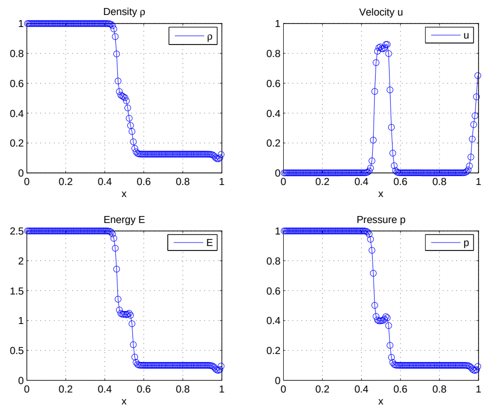

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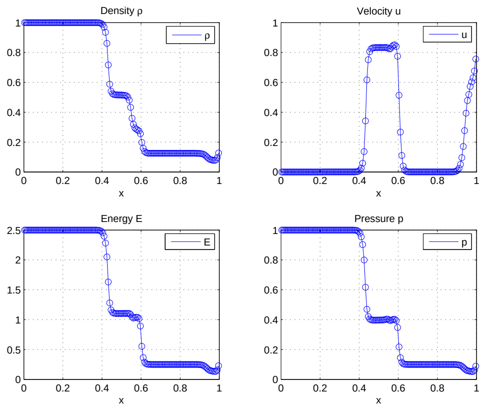
Figure 13: Results of the Davis scheme for for Euler equations (**??**) with the MacCormack scheme and Davis limiter. Here CFL = 0.1. (a) Initial data at $t = 0$. (b) Solution after 50 timesteps.



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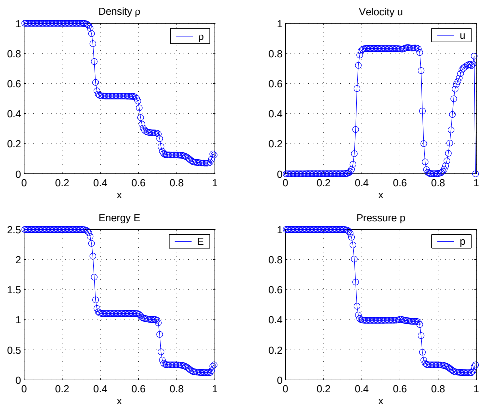

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Figure 14: Results of the Davis scheme for for Euler equations (**??**) with the MacCormack scheme and Davis limiter. Here CFL = 0.1. (c) Solution after 100 timesteps. (d) Solution after 200 timesteps.